Consistent estimation of species abundance from a presence-absence map

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Abstract

The estimation of the abundance of a species using the presence or absence of the species over a grid of cells simplifies data collection but the resulting statistical analysis is challenging. Several estimators have been proposed but their properties are unknown. Here we assume a generalized gamma-Poisson model which allows dependencies across the grid and develop a new estimator for this model. It is shown that this estimator is consistent, allowing us to conclude that it is indeed possible to estimate abundance from presence-absence maps.

Keywords: Presence Absence Map; Consistency; Gamma-Poisson Model.

1 Introduction

In ecological studies where the interest is in estimating the number of individuals of a certain species in a given area it is often too laborious to count all individuals in an area and a more labor effective procedure is to divide the area in M cells and to determine for each cell whether the species is present or not. The data may be thought of as a presence-absence map that identifies the occupied cells and it is often represented by a $I \times J$-matrix $(y_{ij})_{i=1,\ldots,I,\, j=1,\ldots,J}$ with $M = I \cdot J$ cells, where $y_{ij} = 1$

if the i, j cell is unoccupied cell and 0 otherwise. Then the question is whether it is possible to estimate the total number N of the species in the area from these data in general. The analysis of these data presents an interesting statistical challenge (Kunin 1998, He and Gaston 2000, Kunin et al. 2000, Conlinsk et al. 2007, He and Gaston 2007, Hwang and He 2010).

The resolution of the problem requires distributional assumptions on the unknown numbers X_{ij} of individuals in each cell. The simplest approach is to assume that the probability for the occurrence of each of the N individuals is the same for each cell. Then $(X_{ij})_{i=1,\dots,I,\, j=1,\dots,J}$ has a multinomial distribution where each cell has the same probability. Then $N = \sum_{i=1}^{I} \sum_{j=1}^{J} X_{ij}$ can be estimated using maximum likelihood. Alternatively, the X_{ij} may be supposed to be independently and identically distributed with a common Poisson distribution. Then N can be estimated using the method of moments (He and Gaston 2000), however, the resulting estimator is negatively biased and hence not used. Noting the well known relationships between multinomial, Poisson and negative binomial distributions, He and Gaston (2000) proposed the use of the negative binomial distribution. However, as Conlisk et al. (2007) remarked, this leads to an underdetermined problem. It is caused by the fact that the negative binomial distribution has two unknown parameters including the clumping or aggregation parameter k (Pielou 1977). If k is known, then N can be estimated as a function of m_0, the number of empty cells (He and Gaston 2000). Since k is usually not known, it must also be estimated. He and Gaston (2000) used a coarser map, where adjacent pairs of cells are merged, to estimate k. However, as noted by Conlisk et al. (2007), the resulting equations for the two parameters cannot be simultaneously solved and in the negative binomial framework there is no simple solution to this problem (Conlisk et al. 2007, He and Gaston 2007). To date, no statistical properties of the estimators of He and Gaston (2000) are known. Even consistency has not been proven.

Recall that the negative binomial distribution arises from a gamma-Poisson mixture. That is, X_{ij} has a Poisson distribution with mean λ_{ij} which is the realization of a gamma random variable, representing for example environmental variation across the study area. In this model the cell counts are unconditionally independent. This is unrealistic in modelling species abundance where we expect correlation between adjacent areas with similar environments. To model this behaviour Hwang and He (2010) proposed the generalized gamma-Poisson mixture model. In this model, a gamma random variable is first associated with each cell, as in the negative binomial model, but correlations between the cells are modelled by allowing the conditional mean of the Poisson counts to be a weighted sum of all gamma random variables. Hwang and He (2010) used gamma approximations to find an estimator. Using census forest data (Condit et al., 1996) they showed empirically that estimators based on the generalized gamma Poisson model gave improved estimators of abundance from occurrence data. However, they did not give any theoretical properties of their estimator.

In Section 2 we derive an alternate estimator for the generalised gamma-Poisson model whose consistency is shown in Section 3. This consistency proof is not straightforward and needs some tricky arguments which are given in the Appendix. The estimator is applied to real data in Section 4. The results are discussed in Section 5.

2 Model and the new estimator

For simplicity, assume that I and J are even. Formally, the generalized gamma-Poisson mixture model is defined as follows. Let $V_{st},\ s,t = 0,-1,1,-2,-2,\ldots$ be i.i.d. gamma distributed random variables with parameters a and b so that $\mathrm{E}(V_{st}) = ab$ and $\mathrm{var}(V_{st}) = ab^2$. The conditional distribution of X_{ij} given $V_{st} = v_{st}$ for $s,t \in \mathbb{Z}$ is taken to be Poisson with parameter $\lambda_{ij} = \sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}\rho^{|i-s|+|j-t|}v_{st}$ where $0 \le \rho < 1$. Here the parameter ρ reflects the strength of spatial correlation across the cells. Note that for $(i,j) \neq (k,l)$, X_{ij} and X_{kl} are conditionally independent given $V_{st} = v_{st},\ s,t \in \mathbb{Z}$.

Since the conditional distribution of X_{ij} given $V_{st} = v_{st},\ s,t \in \mathbb{Z}$, is Poisson with parameter $\lambda_{ij} = \sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}\rho^{|i-s|+|j-t|}\,v_{st}$, we see that

$$\mathrm{E}(X_{ij}) = \sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}\rho^{|i-s|+|j-t|}\,\mathrm{E}(V_{st}) = \sum_{s=-\infty}^{\infty}\rho^{|i-s|}\sum_{t=-\infty}^{\infty}\rho^{|j-t|}\,ab = \left(\frac{1+\rho}{1-\rho}\right)^2 ab,$$

and thus

$$\mathrm{E}\,(N) = \sum_{i=1}^{I}\sum_{j=1}^{J}E(X_{ij}) = I\,J\,\left(\frac{1+\rho}{1-\rho}\right)^2 ab. \tag{1}$$

Once we have estimators $\widehat{a},\ \widehat{b},\ \widehat{\rho}$ of $a,\ b$, and ρ this yields the estimator

$$\widehat{N} = I\,J\left(\frac{1+\widehat{\rho}}{1-\widehat{\rho}}\right)^2\widehat{a}\,\widehat{b}, \tag{2}$$

and our main task is to estimate these parameters using presence-absence data.

Let $Y_{ij} = 1_{\{0\}}(X_{ij})$ be an indicator of whether the cell (i,j) is empty or not. The variables $Y^d_{i(2j-1)} = Y_{i(2j-1)}Y_{i(2j)}$ and $Y^q_{(2i-1)(2j-1)} = Y_{(2i-1)(2j-1)}Y_{(2i-1)(2j)}Y_{(2i)(2j-1)}Y_{(2i)(2j)}$ indicate whether the double and quadruple cells are empty. Our approach is based on the method of moments and to this end we need:

Proposition 1

$$a)\quad E(Y_{ij}) = \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\left(1+b\,\rho^{|i-s|+|j-t|}\right)^{-a},$$

$$b)\quad E(Y_{ij}^{d}) = \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\left(1+b\,\rho^{|i-s|}(\rho^{|j-t|}+\rho^{|j+1-t|})\right)^{-a},$$

$$c)\quad E(Y_{ij}^{q}) = \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\left(1+b\,(\rho^{|i-s|}+\rho^{|i+1-s|})(\rho^{|j-t|}+\rho^{|j+1-t|})\right)^{-a},$$

Define $M_0 = \sum_{i=1}^{I}\sum_{j=1}^{J} Y_{ij}$, $M_0^d = \sum_{i=1}^{I}\sum_{j=1}^{J/2} Y_{i(2j-1)}^d$ and $M_0^q = \sum_{i=1}^{I/2}\sum_{j=1}^{J/2} Y_{(2i-1)(2j-1)}^q$ with realizations m_0, m_0^d, and m_0^q. Then, a, b, and ρ can be estimated by solving

$$\frac{1}{I\,J}\,m_0 = \mathrm{E}_{(a,b,\rho)}\left(\frac{1}{I\,J}M_0\right) = \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\left(1+b\,\rho^{|s|+|t|}\right)^{-a}, \tag{3}$$

$$\frac{2}{I\,J}\,m_0^d = \mathrm{E}_{(a,b,\rho)}\left(\frac{2}{I\,J}M_0^d\right) = \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\left(1+b\,\rho^{|s|}(\rho^{|t|}+\rho^{|t+1|})\right)^{-a},$$

$$\frac{4}{I\,J}\,m_0^q = \mathrm{E}_{(a,b,\rho)}\left(\frac{4}{I\,J}M_0^q\right) = \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\left(1+b\,(\rho^{|s|}+\rho^{|s+1|})(\rho^{|t|}+\rho^{|t+1|})\right)^{-a}.$$

3 Consistency

To establish consistency we let $IJ \to \infty$ so that the number of cells observed increases. As we suppose the occupancy probabilities do not depend on I or J this implies that $N \to \infty$. This corresponds to increasing the size of the map.

Theorem 1 *If $(a,b,\rho) \in (0,\infty)\times(0,\infty)\times[0,1)$, then $\left(\frac{1}{I\,J}\,M_0, \frac{2}{I\,J}\,M_0^d, \frac{4}{I\,J}\,M_0^q\right)^\top$ is a weakly consistent estimator of $\left(E_{(a,b,\rho)}\left(\frac{1}{I\,J}M_0\right), E_{(a,b,\rho)}\left(\frac{2}{I\,J}M_0^d\right), E_{(a,b,\rho)}\left(\frac{4}{I\,J}M_0^q\right)\right)^\top$ as $I\,J \to \infty$.*

The consistency of the estimators $(\widehat{a},\ \widehat{b},\ \widehat{\rho})$ and thus of $\widehat{N}$ follows from the following general proposition, which is similar to those of Amemiya (1973, Lemma 3), White (1980, Lemma 2.2) and White (1981, Theorem 2.1).

Proposition 2 *Let be $\Theta \subset I\!R^L$ compact, $g : \Theta \to I\!R^L$ continuous satisfying $g(\theta) \neq g(\theta')$ for all $\theta, \theta' \in \Theta$ with $\theta \neq \theta'$, and X_N an L dimensional random vector. If X_N is a weakly consistent estimator for $g(\theta)$, then*

$$\widehat{\theta}_N \in \arg\min_{\theta\in\Theta} \|X_N - g(\theta)\|$$

is a weakly consistent estimator of θ.

Since $\mathrm{E}(N)$ defined in (1) is a continuous function of (a, b, ρ), Theorem 1 implies at once:

Theorem 2 *The abundance estimator $\widehat{N}$ given by (2), where $(a,\ b,\ \rho) \in \Theta$ for a compact parameter space $\Theta \subset (0, \infty) \times (0, \infty) \times (0, 1)$ are solutions of (3), satisfies $\widehat{N} - E(N) = o_P(IJ)$.*

Hence we need only to prove Theorem 1. This proof is not straightforward because of the dependence structure and therefore given together with the proof of Proposition 1 in the Appendix.

4 Application to real data

We consider data from a complete census of a tropical rain forest in Barro Colorado Island (BCI), Panama (Condit et al. 1996). The study area was a 50-hectare (500×1000 m) retangle plot referred to as the BCI plot. The tree plot was established in 1980 and has been surveyed six times so far. The data from the 1985 census are used in this study. The data records the exact location and species for each free-standing tree with diameter at breast height (dbh) at least 1 cm in the plot. In the 1985 census, there are 238,018 trees representing 299 species. We illustrate the computation of the estimator on the the species *Faramea occidentalis*. Here $N = 25094$, $m_0 = 7824$, $m_0^d = 1994$ and $m_0^q = 357$. There were $20000 - 7824 = 12176$ occupied cells, giving a lower bound on N. We solved the estimating equations (3) with the products from negative to positive infinity replaced by products from $-T$ to T by minimizing an objective function consisting of the sums of squares of the differences. We considered $\rho = 0.05, 0.06, \ldots, 0.99$ and for each value of ρ we found values of a and b to minimize the objective function. In Figure 1 we plot the value of the minimum and the estimated $\widehat{N}$ as functions of ρ for $T = 60$. The minimum was quite flat as a function of ρ near the minimum, however, a plot of the logarithm of the minimum, not given here, did show a distinct minimum. At this minimum, $\hat{a} = 0.00066$, $\hat{b} = 3.27$, and $\hat{\rho} = 0.92$ and the estimated abundance was $\widehat{N} = 24774$. In the second part of the figure we see that the estimated value of N is also quite flat and is close to the true value for a range of values of ρ.

5 Discussion

We have demonstrated analytically that the new estimator is consistent and conclude that for the mixed gamma-Poisson model it is possible to estimate abundance from presence-absence maps in a quite general setting.

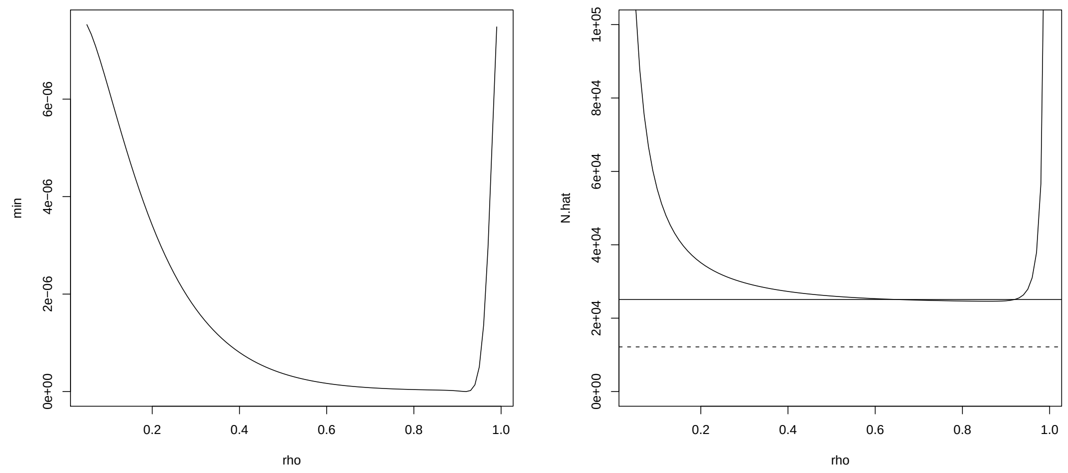

Figure 1: The minimum value of the sums of squared differences of (3) and $\widehat{N}$ as functions of $\rho \in (0, 1)$ for *Faramea occidentalis*. In the second plot, the horizontal line gives the known value of N (solid line) and the lower bound on N (dashed line).

The assumption of a compact parameter space is required to prove Proposition 2 and hence Theorem 2, but since the compact parameter space can be arbitrarily large, this is not a strong restriction. We also require $g(\theta) \neq g(\theta')$ for all $\theta, \theta' \in \Theta$ with $\theta \neq \theta'$. This is an identifiability condition which is usually difficult to verify in nonlinear settings with more than two parameters. Here this means that the solutions $\widehat{a}$, $\widehat{b}$, $\widehat{\rho}$ of the equations given by (3) are unique, i.e. (a, b, ρ) is identified by $\left(\frac{1}{IJ}\, m_0, \frac{2}{IJ}\, m_0^d, \frac{4}{IJ}\, m_0^q\right)^\top$. This is not the case for $\rho = 0$ or $a = 0$ or $b = 0$, but we conjecture that identifiability holds for $\rho > 0$, $a > 0$, $b > 0$. Since $\rho = 0$ does not satisfy the conditions of the Theorem, we have shown consistency only in the case where the cell counts are dependent. Moreover, consistency is obtained as $IJ \to \infty$ with the cell occupancy probabilities remaining the same so that $N \to \infty$. An alternate approach not considered here would be to allow the grid to become finer so that N is the number of individuals in a given area. In this case, the large sample properties are problematic as for a fine enough grid, N will be the total number of occupied cells.

We were not concerned with the practical application of the new estimator but have focused on consistency. In particular we have not addressed the goodness of fit of the model. The precision of the estimator may be addressed using bootstrap methods according to Hall 1988.

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A Proofs

To prove Theorem 1, some propositions must be proved. As Proposition 1 and Proposition 3 below are related we combine their proofs.

Proposition 3

$$a)\qquad E(Y_{ij}\cdot Y_{kl}) = \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\left(1+b\,[\rho^{|i-s|+|j-t|}+\rho^{|k-s|+|l-t|}]\right)^{-a},$$

$$b)\qquad E(Y_{ij}^d\cdot Y_{kl}^d) = \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\left(1+b\,[\rho^{|i-s|}(\rho^{|j-t|}+\rho^{|j+1-t|})+\rho^{|k-s|}(\rho^{|l-t|}+\rho^{|l+1-t|})]\right)^{-a},$$

$$c)\qquad E(Y_{ij}^q\cdot Y_{kl}^q) = \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\left(1+b\,[(\rho^{|i-s|}+\rho^{|i+1-s|})(\rho^{|j-t|}+\rho^{|j+1-t|}) + (\rho^{|k-s|}+\rho^{|k+1-s|})(\rho^{|l-t|}+\rho^{|l+1-t|})]\right)^{-a}.$$

Proof of Propositions 1 and 3.
Since a Poisson random variable X with parameter λ satisfies $P(X=0)=\exp\{-\lambda\}$, using the moment generating function of the gamma distribution and the independence of the V_{st}, we have

$$E(Y_{ij}) = P(X_{ij}=0) = \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\mathrm{E}\left(\exp\{-\rho^{|i-s|+|j-t|}\,V_{st}\}\right) = \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}(1+b\,\rho^{|i-s|+|j-t|})^{-a}$$

which is Proposition 1 a). Since X_{ij} and X_{kl} are conditionally independent given V_{st} with $s,t\in\mathbb{Z}$, Proposition 1 b) and Proposition 3 a) follow from

$$E(Y_{ij}\cdot Y_{kl}) = P(X_{ij}=0, X_{kl}=0) = \mathrm{E}\left(\exp\{-\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(\rho^{|i-s|+|j-t|}+\rho^{|k-s|+|l-t|})\,V_{st}\}\right).$$

Proposition 1 c) is a special case of Proposition 3 b). Since b) and c) of Proposition 3 can be proved similarly, only the proof of c) is shown here:

$$\begin{aligned} E(Y_{ij}^q\cdot Y_{kl}^q) &= P(X_{ij}=0, X_{i(j+1)}=0, X_{(i+1)j}=0, X_{(i+1)(j+1)}=0, \\ &\qquad X_{kl}=0, X_{k(l+1)}=0, X_{(k+1)l}=0, X_{(k+1)(l+1)}=0) \\ &= E\left(\exp\left[-\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}\left(\rho^{|i-s|+|j-t|}+\rho^{|i-s|+|j+1-t|}+\rho^{|i+1-s|+|j-t|}+\rho^{|i+1-s|+|j+1-t|}\right.\right.\right. \\ &\qquad \left.\left.\left. +\rho^{|k-s|+|l-t|}+\rho^{|k-s|+|l+1-t|}+\rho^{|k+1-s|+|l-t|}+\rho^{|k+1-s|+|l+1-t|}\right)V_{st}\right]\right).\square \end{aligned}$$

Proposition 4 *Let be $(a,b,\rho) \in (0,\infty)\times(0,\infty)\times[0,1)$. Then there exists a constant c only depending on a, b, ρ such that*

$$\sum_{i=1}^{I}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{j=1}^{J} \ln\left(\frac{E(Y_{ij}Y_{kl})}{E(Y_{ij})\,E(Y_{kl})}\right) \le c\ I\ J,$$

$$\sum_{i=1}^{I}\sum_{i=1}^{I}\sum_{j=1}^{J/2}\sum_{j=1}^{J/2} \ln\left(\frac{E(Y^d_{i(2j-1)}Y^d_{k(2l-1)})}{E(Y^d_{i(2j-1)})\,E(Y^d_{k(2l-1)})}\right) \le c\ I\ J,$$

$$\sum_{i=1}^{I/2}\sum_{i=1}^{I/2}\sum_{j=1}^{J/2}\sum_{j=1}^{J/2} \ln\left(\frac{E(Y^q_{(2i-1)(2j-1)}Y^q_{(2k-1)(2l-1)})}{E(Y^q_{(2i-1)(2j-1)})\,E(Y^q_{(2k-1)(2l-1)})}\right) \le c\ I\ J.$$

To prove Proposition 4, we need

Lemma 1 *Let $\alpha_i(s) = \rho^{|i-s|} + \rho^{|i+1-s|}$ for $i = 1,\ldots,I$. Then*

$$\sum_{i=1}^{I}\sum_{k=1}^{I}\sum_{s=-\infty}^{\infty} \alpha_i(s)\alpha_k(s) \le \begin{cases} 12\,I, & \text{for } \rho = 0,\\ \frac{1}{(1-\rho^2)(1-\rho)^4}\left(4\cdot 42\,I + \frac{4}{\rho}\,20\,I\right), & \text{for } \rho \in (0,1), \end{cases}$$

for all $I \in \mathbb{N}$.

Proof.
If $\rho \in (0,1)$, then $\alpha_i(s) = \rho^{i-s} + \rho^{i+1-s} = \rho^{i-s}(1+\rho) \le 2\,\rho^{i-s}$ for $s \le i$ and $\alpha_i(s) = \rho^{s-i} + \rho^{s-(i+1)} = \rho^{s-i-1}(\rho+1) \le 2\,\rho^{s-i-1}$ for $s \ge i+1$. Analogous bounds hold for $\alpha_k(s)$. Let $i \le k$. Then

$$\begin{aligned}
&\sum_{s=-\infty}^{\infty} \alpha_i(s)\alpha_k(s)\\
&\le \sum_{s=-\infty}^{i} (2\,\rho^{i-s})\,(2\,\rho^{k-s}) + \sum_{s=i+1}^{k} (2\,\rho^{s-i-1})\,(2\,\rho^{k-s}) + \sum_{s=k+1}^{\infty} (2\,\rho^{s-i-1})\,(2\,\rho^{s-k-1})\\
&= 4\left[\rho^{k-i}\sum_{s=0}^{\infty}(\rho^2)^s \;+\; \rho^{k-i-1}\,(k-i) \;+\; \rho^{k-i}\sum_{s=0}^{\infty}(\rho^2)^s\right]\\
&= 4\left[\rho^{k-i}\,\frac{1}{1-\rho^2} \;+\; \rho^{k-i-1}\,(k-i) \;+\; \rho^{k-i}\,\frac{1}{1-\rho^2}\right]\\
&\le 4\,\frac{\rho^{|k-i|-1}}{1-\rho^2}\,(2\rho + |k-i|)\,.
\end{aligned}$$

Similarly, the same upper bound holds for $i \ge k$. As

$$\sum_{i=1}^{I}\sum_{k=1}^{I} \rho^{|k-i|}\,|k-i| = \frac{2}{(1-\rho)^4}\big((I-1)\rho - 2I\rho^2 + (I+1)\rho^3 + (I+1)\rho^{I+1} - 2I\rho^{I+2} + (I-1)\rho^{I+3}\big) \le \frac{2}{(1-\rho)^4}\,10I,$$

we see that

$$
\begin{aligned}
\sum_{i=1}^{I}\sum_{k=1}^{I}\sum_{s=-\infty}^{\infty}\alpha_i(s)\alpha_k(s) \;&\le\; \sum_{i=1}^{I}\sum_{k=1}^{I} 4\,\frac{\rho^{|k-i|-1}}{1-\rho^2}\,\left(2\rho+|k-i|\right) \\
\le\;& \frac{4}{1-\rho^2}\left(2\sum_{i=1}^{I}\sum_{k=1}^{I}\rho^{|k-i|}|k-i| + 2\,I + \frac{1}{\rho}\sum_{i=1}^{I}\sum_{k=1}^{I}\rho^{|k-i|}|k-i|\right) \\
\le\;& \frac{4}{1-\rho^2}\left(2\,\frac{2}{(1-\rho)^4}\,10\,I + 2\,I + \frac{1}{\rho}\,\frac{2}{(1-\rho)^4}\,10\,I\right) \\
\le\;& \frac{1}{(1-\rho^2)(1-\rho)^4}\left(4\cdot 42\,I + \frac{4}{\rho}\,20\,I\right).
\end{aligned}
$$

If $\rho = 0$, we obtain using $0^0 = 1$

$$
\sum_{s=-\infty}^{\infty}\alpha_i(s)\alpha_k(s) \le \begin{cases} 4, & \text{for } |k-i| \le 1, \\ 0, & \text{for } |k-i| > 1. \end{cases} .
$$

as required. □

Proof of Proposition 4.
To complete the proof of Proposition 4, let

$$
\begin{aligned}
\alpha_i^0(s) &= \rho^{|i-s|}, & \alpha_i(s) &= \rho^{|i-s|} + \rho^{|i+1-s|}, \\
\beta_j^0(t) &= \rho^{|j-t|}, & \beta_j(t) &= \rho^{|j-t|} + \rho^{|j+1-t|}.
\end{aligned}
$$

Setting

$$
\begin{aligned}
\alpha_i^1(s) &= \alpha_i^0(s), & \beta_j^1(t) &= \beta_j^0(t), \\
\alpha_i^2(s) &= \alpha_i^0(s), & \beta_j^2(t) &= \beta_j(t), \\
\alpha_i^3(s) &= \alpha_i(s), & \beta_j^3(t) &= \beta_j(t),
\end{aligned}
$$

and

$$
Y_{ij}^1 = Y_{ij},\; Y_{ij}^2 = Y_{ij}^d,\; Y_{ij}^3 = Y_{ij}^q,
$$

Propositions 1 and 3 provide for $m = 1, 2, 3$

$$
\begin{aligned}
E(Y_{ij}^m) &= \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\left(1 + b\;\alpha_i^m(s)\beta_j^m(t)\right)^{-a}, \\
E(Y_{ij}^m Y_{kl}^m) &= \prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty}\left(1 + b\;[\alpha_i^m(s)\beta_j^m(t) + \alpha_k^m(s)\beta_l^m(t)]\right)^{-a}.
\end{aligned}
$$

The inequality $\ln(1+x) \leq x$ for all $x \geq 0$ yields

$$\begin{aligned}
&\ln\left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m)\, E(Y_{kl}^m)}\right) \\
&= \ln\left(\prod_{s=-\infty}^{\infty}\prod_{t=-\infty}^{\infty} \frac{\left(1+b\,[\alpha_i^m(s)\beta_j^m(t)+\alpha_k^m(s)\beta_l^m(t)]\right)^{-a}}{\left(1+b\,\alpha_i^m(s)\beta_j^m(t)\right)^{-a}\;\left(1+b\,\alpha_k^m(s)\beta_l^m(t)\right)^{-a}}\right) \\
&= a\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}\ln\left(\frac{\left(1+b\,\alpha_i^m(s)\beta_j^m(t)\right)\;\left(1+b\,\alpha_k^m(s)\beta_l^m(t)\right)}{1+b\,[\alpha_i^m(s)\beta_j^m(t)+\alpha_k^m(s)\beta_l^m(t)]}\right) \\
&= a\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}\ln\left(1+\frac{b^2\;\alpha_i^m(s)\beta_j^m(t)\;\alpha_k^m(s)\beta_l^m(t)}{1+b\,[\alpha_i^m(s)\beta_j^m(t)+\alpha_k^m(s)\beta_l^m(t)]}\right) \\
&\overset{a>0,b>0,\rho\geq 0}{\leq} a\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}\ln\left(1+b^2\;\alpha_i^m(s)\beta_j^m(t)\;\alpha_k^m(s)\beta_l^m(t)\right) \\
&\leq a\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} b^2\;\alpha_i^m(s)\;\beta_j^m(t)\;\alpha_k^m(s)\;\beta_l^m(t).
\end{aligned}$$

Let

$$\begin{aligned}
&\mathcal{I}^1 = \{i;\ i=1,\ldots,I\},\ \mathcal{J}^1 = \{j;\ j=1,\ldots,J\}, \\
&\mathcal{I}^2 = \{i;\ i=1,\ldots,I\},\ \mathcal{J}^2 = \{j;\ j=2n-1 \text{ with } n=1,\ldots,J/2\}, \\
&\mathcal{I}^3 = \{i;\ i=2n-1 \text{ with } n=1,\ldots,I/2\}, \\
&\qquad\qquad \mathcal{J}^3 = \{j;\ j=2n-1 \text{ with } n=1,\ldots,J/2\}.
\end{aligned}$$

Then for $m=1,2,3$ using Lemma 1, and noting $\alpha_i^m(s) \leq \alpha_i(s)$ for $i=1,\ldots,I$ and $\beta_j^m(t) \leq \beta_j(t)$ for $j=1,\ldots,J$ we see that

$$\begin{aligned}
&\sum_{i\in\mathcal{I}^m}\sum_{j\in\mathcal{J}^m}\sum_{k\in\mathcal{I}^m}\sum_{l\in\mathcal{J}^m}\ln\left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m)\, E(Y_{kl}^m)}\right) \\
&\leq a\sum_{i\in\mathcal{I}^m}\sum_{j\in\mathcal{J}^m}\sum_{k\in\mathcal{I}^m}\sum_{l\in\mathcal{J}^m}\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} b^2\;\alpha_i^m(s)\;\beta_j^m(t)\;\alpha_k^m(s)\;\beta_l^m(t) \\
&\leq a\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{I}\sum_{l=1}^{J}\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} b^2\;\alpha_i(s)\;\beta_j(t)\;\alpha_k(s)\;\beta_l(t)
\end{aligned}$$

$$
\begin{aligned}
&= a\, b^2 \sum_{i=1}^{I} \sum_{k=1}^{I} \sum_{s=-\infty}^{\infty} \alpha_i(s)\, \alpha_k(s) \sum_{j=1}^{J} \sum_{l=1}^{J} \sum_{t=-\infty}^{\infty} \beta_j(t)\, \beta_l(t) \\
&\le \begin{cases} a\, b^2\; 12^2\; I\; J, & \text{for } \rho = 0 \\ a\, b^2\; \left(\frac{1}{(1-\rho^2)(1-\rho)^4} \left(4 \cdot 42 + \frac{4}{\rho} \cdot 20 \right) \right)^2\; I\; J, & \text{for } \rho \in (0,1) \end{cases} \\
&= c\; I\; J,
\end{aligned}
$$

where the constant c depends only on $a,\ b,\ \rho$. □

Proposition 5 *For* $(a, b, \rho) \in (0, \infty) \times (0, \infty) \times [0, 1)$

$$
\begin{aligned}
a) &\quad \operatorname{var}(M_0) \le c\; I\; J, \\
b) &\quad \operatorname{var}(M_0^d) \le c\; I\; J, \\
c) &\quad \operatorname{var}(M_0^q) \le c\; I\; J,
\end{aligned}
$$

where c *is a constant only depending on* a, b, *and* ρ.

Proof of Proposition 5.
Using the same notation as in the proof of Proposition 4, we have

$$
M_0 = \sum_{i \in \mathcal{I}^1} \sum_{j \in \mathcal{J}^1} Y_{ij}^1,\ M_0^d = \sum_{i \in \mathcal{I}^2} \sum_{j \in \mathcal{J}^2} Y_{ij}^2,\ M_0^q = \sum_{i \in \mathcal{I}^3} \sum_{j \in \mathcal{J}^3} Y_{ij}^3.
$$

Hence for $m = 1, 2, 3$ we have only to prove

$$
\operatorname{var}\left(\sum_{i \in \mathcal{I}^m} \sum_{j \in \mathcal{J}^m} Y_{ij}^m \right) \le c\; I\; J.
$$

First note that

$$
E(Y_{ij}^m) = E(Y_{kl}^m) \quad \text{and} \quad \operatorname{cov}(Y_{ij}^m, Y_{kl}^m) \le \operatorname{var}(Y_{ij}^m) = \operatorname{var}(Y_{kl}^m)
$$

for all $i, k = 1, \ldots, I,\ j, l = 1, \ldots, J$, and $m = 1, 2, 3$. This implies

$$
\begin{aligned}
\frac{\operatorname{var}(Y_{11}^m)}{E(Y_{11}^m)^2} &= \frac{\operatorname{var}(Y_{ij}^m)}{E(Y_{ij}^m)^2} \ge \frac{\operatorname{cov}(Y_{ij}^m, Y_{kl}^m)}{E(Y_{ij}^m)^2} \\
&= \frac{E(Y_{ij}^m\, Y_{kl}^m) - E(Y_{ij}^m)\; E(Y_{kl}^m)}{E(Y_{ij}^m)\; E(Y_{kl}^m)} = \frac{E(Y_{ij}^m\, Y_{kl}^m)}{E(Y_{ij}^m)\; E(Y_{kl}^m)} - 1,
\end{aligned}
$$

so that

$$
\frac{E(Y_{ij}^m\, Y_{kl}^m)}{E(Y_{ij}^m)\; E(Y_{kl}^m)} \le \frac{\operatorname{var}(Y_{11}^m)}{E(Y_{11}^m)^2} + 1 =: c_1 \tag{4}
$$

for all $i, k = 1, \ldots, I$, $j, l = 1, \ldots, J$, and $m = 1, 2, 3$. The mean value theorem provides

$$\begin{aligned}\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)} - 1 &= \exp\left(\ln\left(\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)}\right)\right) - \exp(0)\\ &= \exp(\theta_{ijkl})\left(\ln\left(\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)}\right) - 0\right)\end{aligned}$$

with

$$\theta_{ijkl} \in \left[0, \ln\left(\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)}\right)\right] \text{ if } \ln\left(\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)}\right) \geq 0$$

and

$$\theta_{ijkl} \in \left[\ln\left(\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)}\right), 0\right] \text{ if } \ln\left(\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)}\right) < 0.$$

In particular, θ_{ijkl} satisfies because of (4)

$$\theta_{ijkl} \leq \ln\left(\frac{\text{var}(Y_{11}^m)}{E(Y_{11}^m)^2} + 1\right) = \ln(c_1)$$

so that

$$\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)} - 1 \leq \exp(\ln(c_1))\;\ln\left(\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)}\right) = c_1\;\ln\left(\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)}\right).$$

Then Proposition 4 provides

$$\begin{aligned}&\text{var}\left(\sum_{i\in\mathcal{I}^m}\sum_{j\in\mathcal{J}^m} Y_{ij}^m\right)\\ =\;& \sum_{i\in\mathcal{I}^m}\sum_{j\in\mathcal{J}^m}\sum_{k\in\mathcal{I}^m}\sum_{l\in\mathcal{J}^m} \text{cov}(Y_{ij}^m, Y_{kl}^m)\\ =\;& \sum_{i\in\mathcal{I}^m}\sum_{j\in\mathcal{J}^m}\sum_{k\in\mathcal{I}^m}\sum_{l\in\mathcal{J}^m} \frac{E(Y_{ij}^m\,Y_{kl}^m) - E(Y_{ij}^m)\;E(Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)}\,E(Y_{ij}^m)\;E(Y_{kl}^m)\\ =\;& \sum_{i\in\mathcal{I}^m}\sum_{j\in\mathcal{J}^m}\sum_{k\in\mathcal{I}^m}\sum_{l\in\mathcal{J}^m}\left(\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)} - 1\right) E(Y_{11}^m)^2\\ \leq\;& \sum_{i\in\mathcal{I}^m}\sum_{j\in\mathcal{J}^m}\sum_{k\in\mathcal{I}^m}\sum_{l\in\mathcal{J}^m} c_1\;\ln\left(\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)}\right) E(Y_{11}^m)^2\\ =\;& c_1\,E(Y_{11}^m)^2 \sum_{i\in\mathcal{I}^m}\sum_{j\in\mathcal{J}^m}\sum_{k\in\mathcal{I}^m}\sum_{l\in\mathcal{J}^m} \ln\left(\frac{E(Y_{ij}^m\,Y_{kl}^m)}{E(Y_{ij}^m)\;E(Y_{kl}^m)}\right)\\ \leq\;& c_1\,E(Y_{11}^m)^2\;c_2\;I\;J\\ =\;& c\;I\;J,\end{aligned}$$

where as c_1, $E(Y_{11}^m)^2$, and c_2 only depend on a, b, and ρ so does c. □

Proof of Theorem 1.

Theorem 1 now follows from Proposition 5 a), b) and c) and Chebyshev's inequality. □