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Multivariate sign depth and related distribution-free tests for model fit

Supplementary Material

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Abstract This document complements the article *Multivariate sign depth and related distribution-free tests for model fit.* It contains:

- Section S.1 with detailed proofs of Theorems 3 and 4, and Lemma 5, p. 1;
- Section S.2 with alternative proofs of Theorem 2 (a), (b), (c), and (d), p. 9;
- Section S.3 applying the bivariate simplex depth notions to testing, p. 23;
- Section S.4 applying the bivariate component depths to testing, p. 24; and
- Section S.5 with an explanation of the simulation results for the regression models, p. 27.

S.1 Detailed proofs of Theorems 3 and 4, and Lemma 5

Proof of Theorem 3. The proof is based on the limit theorem of Hoeffding and Robbins (1948) for m-dependent random variables. Hoeffding and Robbins (1948) define random variables $X_1, X_2, \ldots, X_N$ as m-dependent if and only if $(X_1, \ldots, X_r)$ and $(X_s, \ldots, X_N)$ are independent for all $s - r > m$ and prove the asymptotic normal distribution of $\frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n$ under some conditions. In the case of identically distributed random variables, these conditions are

$$\mathbb{E}(X_1) = 0, \ \mathbb{E}(|X_1|^3) < \infty.$$

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Setting

$$A := \mathbb{E}(X_1^2) + 2 \cdot \sum_{d=2}^{m+1} \mathbb{E}(X_1 \cdot X_d),$$

then the limit theorem of Hoeffding and Robbins (1948) provides

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, A) \quad \text{or} \quad \sqrt{N} \frac{\frac{1}{N}\sum_{n=1}^{N} X_n}{\sqrt{A}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \tag{S.1}$$

respectively.

Proof of Theorem 3 (a). Set $V_n := \mathbb{1}\{0_2 \in \mathbb{S}(R_n, R_{n+1}, R_{n+2})\}$ and $X_n := V_n - \frac{1}{4}$. According to Theorem 2 (a), we have $\mathbb{E}(X_n) = 0$. Obviously, $\mathbb{E}(|X_1|^3) < \infty$ is satisfied for an indicator variable. Moreover, Dyckerhoff et al. (2015) already showed that $V_n = \mathbb{1}\{0_2 \in \mathbb{S}(R_n, R_{n+1}, R_{n+2})\}$ and $V_{n+2} = \mathbb{1}\{0_2 \in \mathbb{S}(R_{n+2}, R_{n+3}, R_{n+4})\}$ are stochastically independent so that $(V_1, \ldots, V_r)$ and $(V_{r+2}, \ldots, V_{N-2})$ are independent for all $r = 1, \ldots, N-4$, and thus $X_1, \ldots, X_{N-2}$ are 1-dependent. Since V_n is an indicator variable, we get

$$\mathbb{E}(X_1^2) = \mathbb{E}(V_1^2) - \left(\frac{1}{4}\right)^2 = \mathbb{E}(V_1) - \left(\frac{1}{4}\right)^2 = \frac{1}{4} - \left(\frac{1}{4}\right)^2 = \frac{1}{4} \cdot \frac{3}{4}.$$

Theorem 2 (b) yields

$$\begin{aligned} \mathbb{E}(X_1 \cdot X_2) &= \mathbb{E}(V_1 \cdot V_2) - \left(\frac{1}{4}\right)^2 \\ &= \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1, R_2, R_3) \cap \mathbb{S}(R_2, R_3, R_4)\}) - \left(\frac{1}{4}\right)^2 \\ &= \frac{1}{12} - \left(\frac{1}{4}\right)^2 = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{4} \cdot \frac{1}{12} \end{aligned}$$

and thus

$$\begin{aligned} A = \mathbb{E}(X_1^2) + 2 \cdot \mathbb{E}(X_1 \cdot X_2) &= \frac{1}{4} \cdot \frac{3}{4} + 2 \cdot \frac{1}{4} \cdot \frac{1}{12} \\ &= \frac{1}{4}\left(\frac{3}{4} + 2 \cdot \frac{1}{12}\right) = \frac{1}{4} \cdot \frac{11}{12} = \left(\frac{1}{4}\right)^2 \cdot \frac{11}{3}. \end{aligned}$$

Hence, with $d_1(R_1, \ldots, R_N) - \frac{1}{4} = \frac{1}{N-2} \sum_{n=1}^{N-2} X_n$, the limit theorem of Hoeffding and Robbins (1948) in (S.1) implies

$$\sqrt{N-2} \frac{\frac{1}{N-2} \sum_{n=1}^{N-2} X_n}{\sqrt{A}} = \sqrt{N-2} \ \frac{d_1(R_1, \ldots, R_N) - \frac{1}{4}}{\frac{1}{4} \cdot \sqrt{\frac{11}{3}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Proof of Theorem 3 (b). Set

$$V_n := \mathbb{1}\{0_2 \in \mathbb{S}(R_n, R_{n+1}, R_{n+2}) \cap \mathbb{S}(R_{n+1}, R_{n+2}, R_{n+3})\}$$

and $X_n := V_n - \frac{1}{12}$. According to Theorem 2 (b), we have $\mathbb{E}(X_n) = 0$. Theorem 2 (e) yields that $V_n := \mathbb{1}\{0_2 \in \mathbb{S}(R_n, R_{n+1}, R_{n+2}) \cap \mathbb{S}(R_{n+1}, R_{n+2}, R_{n+3})\}$ and $V_{n+3} := \mathbb{1}\{0_2 \in \mathbb{S}(R_{n+3}, R_{n+4}, R_{n+5}) \cap \mathbb{S}(R_{n+4}, R_{n+5}, R_{n+6})\}$ are stochastically independent so that $(V_1, \ldots, V_r)$ and $(V_{r+3}, \ldots, V_{N-3})$ are independent for all $r = 1, \ldots, N-6$, and thus $X_1, \ldots, X_{N-3}$ are 2-dependent. Again, since V_n is an indicator variable, we get

$$\mathbb{E}(X_1^2) = \mathbb{E}(V_1^2) - \left(\frac{1}{12}\right)^2 = \mathbb{E}(V_1) - \left(\frac{1}{12}\right)^2 = \frac{1}{12} - \left(\frac{1}{12}\right)^2 = \frac{1}{12} \cdot \frac{11}{12}.$$

Theorem 2 (c) yields

$$\begin{aligned}
\mathbb{E}(X_1 \cdot X_2) &= \mathbb{E}(V_1 \cdot V_2) - \left(\frac{1}{12}\right)^2 \\
&= \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1, R_2, R_3) \cap \mathbb{S}(R_2, R_3, R_4)\} \\
&\qquad \cdot \mathbb{1}\{0_2 \in \mathbb{S}(R_2, R_3, R_4) \cap \mathbb{S}(R_3, R_4, R_5)\}) - \left(\frac{1}{12}\right)^2 \\
&= \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1, R_2, R_3) \cap \mathbb{S}(R_2, R_3, R_4) \cap \mathbb{S}(R_3, R_4, R_5)\}) - \left(\frac{1}{12}\right)^2 \\
&= \frac{1}{2^4} \frac{5}{12} - \left(\frac{1}{12}\right)^2 = \frac{1}{12}\left(\frac{5}{16} - \frac{1}{12}\right) = \frac{1}{12} \cdot \frac{15-4}{4 \cdot 4 \cdot 3} = \left(\frac{1}{12}\right)^2 \frac{11}{4}.
\end{aligned}$$

Theorem 2 (d) provides

$$\begin{aligned}
\mathbb{E}(X_1 \cdot X_3) &= \mathbb{E}(V_1 \cdot V_3) - \left(\frac{1}{12}\right)^2 \\
&= \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1, R_2, R_3) \cap \mathbb{S}(R_2, R_3, R_4)\} \\
&\qquad \cdot \mathbb{1}\{0_2 \in \mathbb{S}(R_3, R_4, R_5) \cap \mathbb{S}(R_4, R_5, R_6)\}) - \left(\frac{1}{12}\right)^2 \\
&= \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1, R_2, R_3) \cap \mathbb{S}(R_2, R_3, R_4) \\
&\qquad\quad \cap\ \mathbb{S}(R_3, R_4, R_5) \cap \mathbb{S}(R_4, R_5, R_6)\}) - \left(\frac{1}{12}\right)^2 \\
&= \frac{1}{2^5} \frac{4}{15} - \left(\frac{1}{12}\right)^2 = \frac{4}{2^5 \cdot 3 \cdot 5} - \frac{1}{3 \cdot 3 \cdot 4 \cdot 4} = \frac{1}{3 \cdot 4 \cdot 4}\left(\frac{4}{2 \cdot 5} - \frac{1}{3}\right) \\
&= \frac{1}{3 \cdot 4 \cdot 4}\left(\frac{2}{5} - \frac{1}{3}\right) = \frac{1}{3 \cdot 4 \cdot 4} \frac{6-5}{3 \cdot 5} = \left(\frac{1}{12}\right)^2 \frac{1}{5}.
\end{aligned}$$

This leads to

$$\begin{aligned}
A &= \mathbb{E}(X_1^2) + 2 \cdot \mathbb{E}(X_1 \cdot X_2) + 2 \cdot \mathbb{E}(X_1 \cdot X_3) \\
&= \left(\frac{1}{12}\right)^2 \left(11 + 2 \cdot \frac{11}{4} + 2 \cdot \frac{1}{5}\right) = \left(\frac{1}{12}\right)^2 \frac{220 + 110 + 8}{4 \cdot 5} = \left(\frac{1}{12}\right)^2 \frac{338}{20} \\
&= \left(\frac{1}{12}\right)^2 \frac{169}{10}.
\end{aligned}$$

Hence, with $d_2(R_1,\ldots,R_N)-\frac{1}{12}=\frac{1}{N-3}\sum_{n=1}^{N-3}X_n$, the limit theorem of Hoeffding and Robbins (1948) in (S.1) implies

$$\sqrt{N-3}\,\frac{\frac{1}{N-3}\sum_{n=1}^{N-3}X_n}{\sqrt{A}}=\sqrt{N-3}\;\frac{d_2(R_1,\ldots,R_N)-\frac{1}{12}}{\frac{1}{12}\cdot\sqrt{\frac{169}{10}}}\xrightarrow{\mathcal{D}}\mathcal{N}(0,1)$$

that we wanted to prove. □

Proof of Theorem 4. We show only (b) since the proof for (a) is similar. Set $Y_{n_1,n_2,n_3,n_4}=\mathbb{1}\{0_2\in\mathbb{S}(R_{n_1},R_{n_2},R_{n_3})\cap\mathbb{S}(R_{n_2},R_{n_3},R_{n_4})\}-\frac{1}{12}$. Then $|Y_{n_1,n_2,n_3,n_4}|\le 1$ and $\mathbb{E}(Y_{n_1,n_2,n_3,n_4})=0$ according to Theorem 2 (b). Moreover, according to Theorem 2 (f),

$$\mathbb{E}(Y_{n_1,n_2,n_3,n_4}\cdot Y_{m_1,m_2,m_3,m_4})=\mathbb{E}(Y_{n_1,n_2,n_3,n_4})\cdot\mathbb{E}(Y_{m_1,m_2,m_3,m_4})=0$$

if $\sharp\left(\{n_1,n_2,n_3,n_4\}\cap\{m_1,m_2,m_3,m_4\}\right)\le 1$, where $\sharp$ denotes the number of elements of a set. Then we get

$$\begin{aligned}
&\operatorname{Var}\left(N\left(d_2^F(R_1,\ldots,R_N)-\frac{1}{12}\right)\right)\\
&=\operatorname{Var}\left(\frac{N}{\binom{N}{4}}\sum_{1\le n_1<n_2<n_3<n_4\le N}Y_{n_1,n_2,n_3,n_4}\right)\\
&=\left(\frac{N}{\binom{N}{4}}\right)^2\mathbb{E}\left(\left(\sum_{1\le n_1<n_2<n_3<n_4\le N}Y_{n_1,n_2,n_3,n_4}\right)^2\right)\\
&=\left(\frac{N}{\binom{N}{4}}\right)^2\mathbb{E}\left(\sum_{\substack{1\le n_1<n_2<n_3<n_4\le N\\1\le m_1<m_2<m_3<m_4\le N}}Y_{n_1,n_2,n_3,n_4}\cdot Y_{m_1,m_2,m_3,m_4}\right)\\
&=\left(\frac{N}{\binom{N}{4}}\right)^2\sum_{\substack{1\le n_1<n_2<n_3<n_4\le N\\1\le m_1<m_2<m_3<m_4\le N\\ \sharp\left(\{n_1,n_2,n_3,n_4\}\cap\{m_1,m_2,m_3,m_4\}\right)>1}}\mathbb{E}(Y_{n_1,n_2,n_3,n_4}\cdot Y_{m_1,m_2,m_3,m_4})\\
&\le\left(\frac{N}{\binom{N}{4}}\right)^2\left(\binom{N}{4}+\binom{N}{5}\binom{5}{4}\binom{4}{3}+\binom{N}{6}\binom{6}{4}\binom{4}{2}\right)\\
&\le c_1\,\frac{1}{(N-1)^2(N-2)^2(N-3)^2}\,N^6\le c,
\end{aligned}$$

as desired. □

Proof of Lemma 5. According to Lemma 3, we can assume that all A_n have a uniform distribution on $[0, 1]$. Moreover, according to Lemma 4, we have

$$\begin{aligned}\psi_1(a_2, a_3) &:= \mathbb{P}\Big(\big\{0_2 \in \mathbb{S}(R_1, R_2, R_3)\big\}\big|A_2 = a_2,\ A_3 = a_3\Big)\\ &= \min\left(|a_2 - a_3|\,, 1 - |a_2 - a_3|\right).\end{aligned}$$

Part (a). Since R_1 and R_4 are independent and identically distributed, we get

$$\begin{aligned}&\mathbb{P}\Big(\big\{0_2 \in \mathbb{S}(R_1, R_2, R_3) \cap \mathbb{S}(R_2, R_3, R_4)\big\}\big|A_2 = a_2\Big)\\ &= \int_0^1 \mathbb{P}\Big(\big\{0_2 \in \mathbb{S}(R_1, R_2, R_3) \cap \mathbb{S}(R_2, R_3, R_4)\big\}\big|A_2 = a_2,\ A_3 = a_3\Big)\,\mathrm{d}\,a_3\\ &= \int_0^1 \mathbb{P}\Big(\big\{0_2 \in \mathbb{S}(R_1, R_2, R_3)\big\}\big|A_2 = a_2,\ A_3 = a_3\Big)\\ &\qquad\qquad \cdot \mathbb{P}\Big(\big\{0_2 \in \mathbb{S}(R_2, R_3, R_4)\big\}\big|A_2 = a_2,\ A_3 = a_3\Big)\,\mathrm{d}\,a_3\\ &= \int_0^1 \psi_1(a_2, a_3)^2\,\mathrm{d}\,a_3.\end{aligned}$$

For $a_2 \in [0, \frac{1}{2}]$, we get

$$\begin{aligned}&\int_0^1 \psi_1(a_2, a_3)^2\,\mathrm{d}\,a_3\\ &= \int_0^{\frac{1}{2}} |a_2 - a_3|^2\,\mathrm{d}\,a_3 + \int_{\frac{1}{2}, a_3 - a_2 < \frac{1}{2}}^1 |a_2 - a_3|^2\,\mathrm{d}\,a_3\\ &\quad + \int_{\frac{1}{2}, a_3 - a_2 > \frac{1}{2}}^1 (1 - |a_2 - a_3|)^2\,\mathrm{d}\,a_3\\ &= \int_0^{\frac{1}{2}} (a_3 - a_2)^2\,\mathrm{d}\,a_3 + \int_{\frac{1}{2}}^{\frac{1}{2}+a_2} (a_3 - a_2)^2\,\mathrm{d}\,a_3\\ &\quad + \int_{\frac{1}{2}+a_2}^1 \left(1 - 2\,(a_3 - a_2) + (a_3 - a_2)^2\right)\mathrm{d}\,a_3\\ &= \int_0^1 (a_3 - a_2)^2\,\mathrm{d}\,a_3 + \int_{\frac{1}{2}+a_2}^1 \left(1 - 2\,(a_3 - a_2)\right)\mathrm{d}\,a_3\\ &= \frac{1}{3}(a_3 - a_2)^3\Big|_0^1 + \left(a_3 - 2\,\frac{1}{2}\,(a_3 - a_2)^2\right)\Big|_{\frac{1}{2}+a_2}^1\\ &= \frac{1}{3}(1 - a_2)^3 - \frac{1}{3}a_2^3 + \Big(1 - 2\,\frac{1}{2}\,(1 - a_2)^2\Big) - \Big(\frac{1}{2} + a_2 - 2\,\frac{1}{2}\,\big(\frac{1}{2} + a_2 - a_2\big)^2\Big)\\ &= \frac{1}{3}(1 - a_2)^3 - \frac{1}{3}a_2^3 + \Big(1 - (1 - a_2)^2\Big) - \Big(\frac{1}{2} + a_2 - \big(\frac{1}{2}\big)^2\Big)\\ &= \frac{1}{3}(1 - 3a_2 + 3a_2^2 + a_2^3) - \frac{1}{3}a_2^3 + \Big(1 - (1 - 2a_2 + a_2^2)\Big) - \Big(\frac{1}{4} + a_2\Big)\\ &= \frac{1}{3} - a_2 + a_2^2 + 2a_2 - a_2^2 - \frac{1}{4} - a_2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.\end{aligned}$$

Because of symmetry, the same holds for $a_2 \in [\frac{1}{2}, 1]$.

Part (b). This part of the proof is more complicated. Because of the relationship between R_n and A_n, we get

$$\begin{aligned}
&\mathbb{P}\Big(\big\{0_2 \in \mathbb{S}(R_1,R_2,R_3)\cap \mathbb{S}(R_2,R_3,R_4)\big\}\big|A_1=a_1\Big)\\
&= \int_0^1\int_0^1 \mathbb{P}\Big(\big\{0_2 \in \mathbb{S}(R_1,R_2,R_3)\cap \mathbb{S}(R_2,R_3,R_4)\big\}\big|A_1=a_1,\\
&\qquad\qquad\qquad\qquad A_2=a_2,\ A_3=a_3\Big)\,\mathrm{d}\,a_2\,\mathrm{d}\,a_3\\
&= \int_0^1\int_0^1 \mathbb{1}\big\{0_2\in\mathbb{S}(r_1,r_2,r_3)\big\}\\
&\qquad\qquad \cdot\mathbb{P}\Big(\big\{0_2 \in \mathbb{S}(R_2,R_3,R_4)\big\}\big|\,A_2=a_2,\ A_3=a_3\Big)\,\mathrm{d}\,a_2\,\mathrm{d}\,a_3\\
&= \int_0^1\int_0^1 \mathbb{1}\big\{0_2\in\mathbb{S}(r_1,r_2,r_3)\big\}\cdot\,\psi_1(a_2,a_3)\,\mathrm{d}\,a_2\,\mathrm{d}\,a_3.
\end{aligned}$$

Again consider first $a_1 \in [0, \frac{1}{2}]$. Lemma 2 provides conditions for a_2 and a_3 so that $0_2 \in \mathbb{S}(r_1, r_2, r_3)$ is satisfied. First note that the conditions of Lemma 2 (a) cannot hold for $a_1 \in [0, \frac{1}{2}]$. Hence we have

$$\mathbb{P}\Big(\big\{0_2 \in \mathbb{S}(R_1,R_2,R_3)\cap \mathbb{S}(R_2,R_3,R_4)\big\}\big|A_1=a_1\Big) = I_b + I_c + I_d$$

with

$$I_b := \int_0^1\int_0^1 \mathbb{1}\big\{a_2,\ a_3 \text{ satify conditions of Lemma 2 (b)}\big\}\ \psi_1(a_2,a_3)\,\mathrm{d}\,a_2\,\mathrm{d}\,a_3,$$

$$I_c := \int_0^1\int_0^1 \mathbb{1}\big\{a_2,\ a_3 \text{ satify conditions of Lemma 2 (c)}\big\}\ \psi_1(a_2,a_3)\,\mathrm{d}\,a_2\,\mathrm{d}\,a_3,$$

$$I_d := \int_0^1\int_0^1 \mathbb{1}\big\{a_2,\ a_3 \text{ satify conditions of Lemma 2 (d)}\big\}\ \psi_1(a_2,a_3)\,\mathrm{d}\,a_2\,\mathrm{d}\,a_3.$$

If a_2, a_3 satisfy the conditions of Lemma 2 (b) then $a_2, a_3 \in \big[\frac{1}{2}, 1\big]$ and $\min(a_2, a_3) - \frac{1}{2} < a_1 < \max(a_2, a_3) - \frac{1}{2}$ which is equivalent to

$$\frac{1}{2} \le \min(a_2,a_3) < a_1 + \frac{1}{2} < \max(a_2,a_3) \le 1$$

so that $\psi_1(a_2, a_3) = \max(a_2, a_3) - \min(a_2, a_3)$ and

$$\begin{aligned}
&I_b\\
&= \int_{a_1+\frac{1}{2}}^1 \int_{\frac{1}{2}}^{a_1+\frac{1}{2}} (a_3-a_2)\,\mathrm{d}\,a_2\,\mathrm{d}\,a_3 + \int_{a_1+\frac{1}{2}}^1 \int_{\frac{1}{2}}^{a_1+\frac{1}{2}} (a_2-a_3)\,\mathrm{d}\,a_3\,\mathrm{d}\,a_2
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_{a_1+\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{a_1+\frac{1}{2}}(a_3-a_2)\,\mathrm{d}\,a_2\,\mathrm{d}\,a_3 = 2\int_{a_1+\frac{1}{2}}^{1}(a_3\,a_2-\frac{1}{2}a_2^2)\Big|_{\frac{1}{2}}^{a_1+\frac{1}{2}}\,\mathrm{d}\,a_3\\
&= 2\int_{a_1+\frac{1}{2}}^{1}\left(a_3\,(a_1+\frac{1}{2})-\frac{1}{2}(a_1+\frac{1}{2})^2\right)-\left(a_3\,\frac{1}{2}-\frac{1}{2}(\frac{1}{2})^2\right)\,\mathrm{d}\,a_3\\
&= 2\int_{a_1+\frac{1}{2}}^{1}\left(a_3\,a_1+a_3\,\frac{1}{2}-\frac{1}{2}(a_1^2+a_1+\frac{1}{4})-a_3\,\frac{1}{2}+\frac{1}{8}\right)\,\mathrm{d}\,a_3\\
&= 2\int_{a_1+\frac{1}{2}}^{1}\left(a_3\,a_1-\frac{1}{2}(a_1^2+a_1)\right)\,\mathrm{d}\,a_3\\
&= 2\left(\frac{1}{2}a_3^2\,a_1-a_3\,\frac{1}{2}(a_1^2+a_1)\right)\Big|_{a_1+\frac{1}{2}}^{1}\\
&= 2\left(\frac{1}{2}\,a_1-\frac{1}{2}(a_1^2+a_1)-\left[\frac{1}{2}(a_1+\frac{1}{2})^2\,a_1-(a_1+\frac{1}{2})\,\frac{1}{2}(a_1^2+a_1)\right]\right)\\
&= 2\left(-\frac{1}{2}a_1^2-\left[\frac{1}{2}(a_1^2+a_1+\frac{1}{4})\,a_1-\frac{1}{2}(a_1^3+a_1^2)-\frac{1}{4}\,(a_1^2+a_1)\right]\right)\\
&= 2\left(-\frac{1}{2}a_1^2-\left[\frac{1}{2}a_1^3+\frac{1}{2}a_1^2+\frac{1}{8}\,a_1-\frac{1}{2}a_1^3-\frac{1}{2}a_1^2-\frac{1}{4}\,a_1^2-\frac{1}{4}a_1\right]\right)\\
&= 2\left(-\frac{1}{2}a_1^2-\left[-\frac{1}{8}\,a_1-\frac{1}{4}\,a_1^2\right]\right) = 2\left(-\frac{1}{2}a_1^2+\frac{1}{8}\,a_1+\frac{1}{4}\,a_1^2\right)\\
&= 2\left(-\frac{1}{4}a_1^2+\frac{1}{8}\,a_1\right) = -\frac{1}{2}a_1^2+\frac{1}{4}\,a_1.
\end{aligned}
$$

The second equality holds because of symmetry of a_2 and a_3 so that the integrals for $a_2 < a_3$ and $a_3 < a_2$ are the same so that we can consider only the case $a_2 < a_3$.

If a_2, a_3 satisfy the conditions of Lemma 2 (c) then $\min(a_2, a_3) \in \left[0, \frac{1}{2}\right]$, $\max(a_2, a_3) \in \left[\frac{1}{2}, 1\right]$, $\max(a_2, a_3) - \min(a_2, a_3) = |a_2 - a_3| > \frac{1}{2}$, and $\max(a_2, a_3) - \frac{1}{2} < a_1 < \min(a_2, a_3) + \frac{1}{2}$. Noting $a_1 - \frac{1}{2} \leq 0$, this means

$$
\frac{1}{2} \leq \max(a_2, a_3) < a_1 + \frac{1}{2},\ 0 \leq \min(a_2, a_3) < \max(a_2, a_3) - \frac{1}{2},
$$

and $\psi_1(a_2, a_3) = 1 - |a_2 - a_3| = 1 + \min(a_2, a_3) - \max(a_2, a_3)$. With the same argument as above, the integrals for $a_2 < a_3$ and $a_3 < a_2$ are the same so that we get

$$
\begin{aligned}
&I_c\\
&= \int_{\frac{1}{2}}^{a_1+\frac{1}{2}}\int_{0}^{a_3-\frac{1}{2}}(1+a_2-a_3)\,\mathrm{d}\,a_2\,\mathrm{d}\,a_3\\
&\quad + \int_{\frac{1}{2}}^{a_1+\frac{1}{2}}\int_{0}^{a_2-\frac{1}{2}}(1+a_3-a_2)\,\mathrm{d}\,a_3\,\mathrm{d}\,a_2\\
&= 2\int_{\frac{1}{2}}^{a_1+\frac{1}{2}}\int_{0}^{a_3-\frac{1}{2}}(1+a_2-a_3)\,\mathrm{d}\,a_2\,\mathrm{d}\,a_3
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_{\frac{1}{2}}^{a_1+\frac{1}{2}} \left((1-a_3)a_2 + \frac{1}{2}a_2^2\right)\Big|_0^{a_3-\frac{1}{2}} \,\mathrm{d}\,a_3\\
&= 2\int_{\frac{1}{2}}^{a_1+\frac{1}{2}} \left((1-a_3)(a_3-\frac{1}{2}) + \frac{1}{2}(a_3-\frac{1}{2})^2\right) \,\mathrm{d}\,a_3\\
&= 2\int_{\frac{1}{2}}^{a_1+\frac{1}{2}} \left(a_3 - a_3^2 - \frac{1}{2} + \frac{1}{2}a_3 + \frac{1}{2}(a_3^2 - a_3 + \frac{1}{4})\right) \,\mathrm{d}\,a_3\\
&= 2\int_{\frac{1}{2}}^{a_1+\frac{1}{2}} \left(a_3 - \frac{1}{2}a_3^2 - \frac{3}{8}\right) \,\mathrm{d}\,a_3 = \int_{\frac{1}{2}}^{a_1+\frac{1}{2}} \left(2a_3 - a_3^2 - \frac{3}{4}\right) \,\mathrm{d}\,a_3\\
&= \left(2\frac{1}{2}a_3^2 - \frac{1}{3}a_3^3 - \frac{3}{4}a_3\right)\Big|_{\frac{1}{2}}^{a_1+\frac{1}{2}}\\
&= \left((a_1+\frac{1}{2})^2 - \frac{1}{3}(a_1+\frac{1}{2})^3 - \frac{3}{4}(a_1+\frac{1}{2})\right) - \left((\frac{1}{2})^2 - \frac{1}{3}(\frac{1}{2})^3 - \frac{3}{4}(\frac{1}{2})\right)\\
&= a_1^2 + a_1 + \frac{1}{4} - \frac{1}{3}\left(a_1^3 + 3a_1^2\frac{1}{2} + 3a_1\frac{1}{4} + \frac{1}{8}\right) - \frac{3}{4}a_1 - \frac{3}{8}\\
&\quad - \left(\frac{2}{8} - \frac{1}{3\cdot 8} - \frac{3}{8}\right)\\
&= a_1^2 + a_1 - \frac{1}{8} - \frac{1}{3}a_1^3 - \frac{1}{2}a_1^2 - \frac{1}{4}a_1 - \frac{1}{3\cdot 8} - \frac{3}{4}a_1 + \frac{1}{8} + \frac{1}{3\cdot 8}\\
&= \frac{1}{2}a_1^2 - \frac{1}{3}a_1^3.
\end{aligned}
$$

If a_2, a_3 satisfy the conditions of Lemma 2 (d) then $\min(a_2,a_3) \in \left[0,\frac{1}{2}\right]$, $\max(a_2,a_3) \in \left[\frac{1}{2},1\right]$, $\max(a_2,a_3) - \min(a_2,a_3) = |a_2 - a_3| < \frac{1}{2}$, $0 \le a_1 < \max(a_2,a_3) - \frac{1}{2}$, and $\min(a_2,a_3) + \frac{1}{2} < a_1 \le 1$. Noting that $\min(a_2,a_3)+\frac{1}{2} < a_1$ is not possible for $a_1 \in [0,\frac{1}{2}]$, this is equivalent to

$$
a_1 + \frac{1}{2} \le \max(a_2,a_3) \le 1,\ \max(a_2,a_3) - \frac{1}{2} < \min(a_2,a_3) \le \frac{1}{2},
$$

and $\psi_1(a_2,a_3) = |a_2 - a_3| = \max(a_2,a_3) - \min(a_2,a_3)$. With the same argument as above, the integrals for $a_2 < a_3$ and $a_3 < a_2$ are the same so that we consider again only $a_2 < a_3$ and get

$$
\begin{aligned}
&I_d\\
&= 2\int_{a_1+\frac{1}{2}}^{1}\int_{a_3-\frac{1}{2}}^{\frac{1}{2}} (a_3 - a_2)\,\mathrm{d}\,a_2\,\mathrm{d}\,a_3\\
&= 2\int_{a_1+\frac{1}{2}}^{1} (a_3a_2 - \frac{1}{2}a_2^2)\Big|_{a_3-\frac{1}{2}}^{\frac{1}{2}} \,\mathrm{d}\,a_3\\
&= 2\int_{a_1+\frac{1}{2}}^{1} \left(a_3\frac{1}{2} - \frac{1}{2}\frac{1}{4}\right) - \left(a_3(a_3-\frac{1}{2}) - \frac{1}{2}(a_3-\frac{1}{2})^2\right) \,\mathrm{d}\,a_3\\
&= 2\int_{a_1+\frac{1}{2}}^{1} \frac{1}{2}a_3 - \frac{1}{8} - a_3^2 + \frac{1}{2}a_3 + \frac{1}{2}(a_3^2 - a_3 + \frac{1}{4})\,\mathrm{d}\,a_3
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_{a_1+\frac{1}{2}}^{1} \frac{1}{2}a_3 - \frac{1}{2}a_3^2 \,\mathrm{d}\, a_3 = \int_{a_1+\frac{1}{2}}^{1} a_3 - a_3^2 \,\mathrm{d}\, a_3 \\
&= (\frac{1}{2}a_3^2 - \frac{1}{3}a_3^3)\Big|_{a_1+\frac{1}{2}}^{1} \\
&= (\frac{1}{2} - \frac{1}{3}) - \left(\frac{1}{2}(a_1 + \frac{1}{2})^2 - \frac{1}{3}(a_1 + \frac{1}{2})^3\right) \\
&= \frac{1}{6} - \left(\frac{1}{2}(a_1^2 + a_1 + \frac{1}{4}) - \frac{1}{3}(a_1^3 + 3a_1^2\frac{1}{2} + 3a_1\frac{1}{4} + \frac{1}{8})\right) \\
&= \frac{1}{6} - \left(\frac{1}{2}a_1^2 + \frac{1}{2}a_1 + \frac{1}{8} - \frac{1}{3}a_1^3 - a_1^2\frac{1}{2} - a_1\frac{1}{4} - \frac{1}{3\cdot 8}\right) \\
&= \frac{1}{6} - \left(\frac{1}{4}a_1 + \frac{3}{3\cdot 8} - \frac{1}{3}a_1^3 - \frac{1}{3\cdot 8}\right) \\
&= \frac{4}{3\cdot 8} - \frac{1}{4}a_1 - \frac{2}{3\cdot 8} + \frac{1}{3}a_1^3 \\
&= \frac{1}{12} - \frac{1}{4}a_1 + \frac{1}{3}a_1^3.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\mathbb{P}\Big(\big\{0_2 \in \mathbb{S}(R_1, R_2, R_3) \cap \mathbb{S}(R_2, R_3, R_4)\big\}\big|A_1 = a_1\Big) \\
&= I_b + I_c + I_d \\
&= -\frac{1}{2}a_1^2 + \frac{1}{4}\,a_1 + \frac{1}{2}a_1^2 - \frac{1}{3}a_1^3 + \frac{1}{12} - \frac{1}{4}a_1 + \frac{1}{3}a_1^3 = \frac{1}{12}.
\end{aligned}
$$

Because of symmetry, the same holds for $a_1 \in [\frac{1}{2}, 1]$. □

S.2 Alternative proofs of Theorem 2 (a)–(d)

We provide here an alternative proof which extends the approach of Dyckerhoff et al. (2015). In particular the expectations in Theorem 2 (c) and (d) are derived explicitly so that a computer algebra system is not needed.

Lemma S.1 *If r_1, r_2, r_3 are in general position, then there are exactly two vectors $(s_1, s_2, s_3) \in \{-1, 1\}^3$ such that $0_2 \in \mathbb{S}(s_1 r_1, s_2 r_2, s_3 r_3)$ and the two vectors are the opposite of each other, i.e., the second vector is given by $(-s_1, -s_2, -s_3)$.*

Proof. This is Lemma 1 in Dyckerhoff et al. (2015). □

Definition S.1 Define the mapping $s : \mathbb{R}^3 \to \{1\} \times \{-1, 1\}^2$ such that $s(r)$ for all $r \in \mathbb{R}^3$ is the unique vector (s_1, s_2, s_3) of Lemma S.1 with positive first component, i.e., $s_1 = 1$.

Note for $s(r) = (s_1, s_2, s_3)$ that $-s(r) = (-s_1, -s_2, -s_3) = (-1, -s_2, -s_3)$ is then the second unique vector of Lemma S.1.

Classical proof of Theorem 2 (a). Assumptions (A_1) and (A_2) provide

$$\begin{aligned}
&\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1, R_2, R_3)\}) = \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(S_1\, R_1, S_2\, R_2, S_3\, R_3)\}) \\
&= \mathbb{E}(\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(S_1\, R_1, S_2\, R_2, S_3\, R_3)\})|(R_1, R_2, R_3)) \\
&= \int_{(\mathbb{R}^2)^3} \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(S_1\, r_1, S_2\, r_2, S_3\, r_3)\})\, \mathrm{d}\,(\mathbb{P}^{(R_1,R_2,R_3)}(r_1, r_2, r_3)).
\end{aligned}$$

Then Lemma S.1 together with the assumptions (A_4) and (A_5) yield

$$\begin{aligned}
&\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(S_1\, r_1, S_2\, r_2, S_3\, r_3)\}) \\
&= \mathbb{P}\left((S_1, S_2, S_3) = s((r_1, r_2, r_3)) \text{ or } (S_1, S_2, S_3) = -s((r_1, r_2, r_3))\right) \\
&= \frac{2}{2^3} = \frac{1}{4}
\end{aligned}$$

so that

$$\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1, R_2, R_3)\}) = \int_{(\mathbb{R}^2)^3} \frac{1}{4}\, \mathrm{d}\,(\mathbb{P}^{(R_1,R_2,R_3)}(r_1, r_2, r_3)) = \frac{1}{4}.$$

The proof is finished. □

The result of Theorem 2 (a) also can be obtained by a different approach, which leads to the proofs of Theorem 2 (b), (c), and (d). As in the proof of Lemma 2 in Dyckerhoff et al. (2015), we limit ourselves now on $\mathbb{R} \times \mathbb{R}_+$ instead of $\mathbb{R}^2$. The restriction to $\mathbb{R} \times \mathbb{R}_+$ allows to order $(r_1, r_2, \ldots, r_K) \in (\mathbb{R} \times \mathbb{R}_+)^K$ according to the angle between the first positive semi-axis and the halfline $\{\lambda\, r_n;\ \lambda \geq 0\}$, $n = 1, \ldots, K$, which can be expressed by the $\arccos\colon [-1, 1] \to [0, \pi]$ function applied to $r_{n,1}/\|r_n\|$.

Definition S.2 Let $\Pi(1, 2, \ldots, K)$ the set of all permutations of $(1, 2, \ldots, K)$. For $\pi = (\pi(1), \ldots, \pi(K)) \in \Pi(1, 2, \ldots, K)$ define

$$\begin{aligned}
\mathbb{A}_K(\pi) := \{&(r_1, r_2, \ldots, r_K) \in (\mathbb{R} \times \mathbb{R}_+)^K;\ \arccos(r_{\pi(1),1}/\|r_{\pi(1)}\|) \\
&< \arccos(r_{\pi(2),1}/\|r_{\pi(2)}\|) < \ldots < \arccos(r_{\pi(K),1}/\|r_{\pi(K)}\|)\}
\end{aligned}$$

and the equivalence class with representative π

$$\begin{aligned}
E_K(\pi) := \{&(\pi(1), \pi(2), \pi(3), \ldots, \pi(K)),\ (\pi(2), \pi(3), \ldots, \pi(K), \pi(1)), \\
&(\pi(3), \ldots, \pi(K), \pi(1), \pi(2)),\ \ldots,\ (\pi(K), \pi(1), \pi(2), \ldots, \pi(K-1))\}.
\end{aligned}$$

$\mathbb{A}_K(\pi)$ and $E_K(\pi)$ are defined analogously if $\pi \in \Pi(l+1, l+2, \ldots, l+K)$ with $l \in \mathbb{N}$, where $\Pi(l+1, l+2, \ldots, l+K)$ is the set of all permutations of $(l+1, l+2, \ldots, l+K)$.

Note that assumption (A_5) provides

$$\sum_{\pi\in\Pi(1,2,\ldots,K)} \mathbb{P}^{R_1,\ldots,R_K}\left(\mathbb{A}_K(\pi)\right) \tag{S.2}$$
$$= \mathbb{P}^{R_1,\ldots,R_K}\left(\bigcup_{\pi\in\Pi(1,2,\ldots,K)} (\mathbb{A}_K(\pi))\right) = \mathbb{P}^{R_1,\ldots,R_K}((\mathbb{R}\times\mathbb{R}_+)^K).$$

Since, according to (A_1)–(A_4), we have

$$\mathbb{P}^{R_1,\ldots,R_K}\left((\mathbb{R}\times\mathbb{R}_+)^K\right) = \frac{1}{2^K}$$

we get with (S.2) and $\sharp\,(\Pi(1,2,\ldots,K)) = K!$, where $\sharp$ denotes the cardinality of a set,

$$\mathbb{P}^{R_1,\ldots,R_K}(\mathbb{A}_K(\pi)) = \frac{1}{2^K}\,\frac{1}{K!}. \tag{S.3}$$

Lemma S.2 *Let s be the mapping defined in Definition S.1, $\pi \in \Pi(1,2,3)$, and $r \in \mathbb{A}_3(\pi)$. Then $s((r_{\pi(1)}, r_{\pi(2)}, r_{\pi(3)})) = (1,-1,1)$ and $(s_1,s_2,s_3) = s(r)$ satisfies $(s_{\pi(1)}, s_{\pi(2)}, s_{\pi(3)}) = (1,-1,1)$ or $(s_{\pi(1)}, s_{\pi(2)}, s_{\pi(3)}) = (-1,1,-1)$.*

Proof. Since $\arccos(r_{\pi(1),1}/\|r_{\pi(1)}\|) < \arccos(r_{\pi(2),1}/\|r_{\pi(2)}\|) < \arccos(r_{\pi(3),1}/\|r_{\pi(3)}\|)$, it holds $0_2 \in \mathbb{S}(r_{\pi(1)}, -r_{\pi(2)}, r_{\pi(3)})$ so that $s((r_{\pi(1)}, r_{\pi(2)}, r_{\pi(3)})) = (1,-1,1)$. Since the first component s_1 of s should be positive, we have to distinguish between two cases. Case 1: $\pi(1) = 1$ or $\pi(3) = 1$ then $(s_{\pi(1)}, s_{\pi(2)}, s_{\pi(3)}) = (1,-1,1)$. Case 2: $\pi(2) = 1$ then $(s_{\pi(1)}, s_{\pi(2)}, s_{\pi(3)}) = (-1,1,-1)$.

Alternative proof of Theorem 2 (a). As in the first proof, we get with the assumptions (A_1) and (A_2)

$$\begin{aligned}
&\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1,R_2,R_3)\})\\
&= \int_{(\mathbb{R}^2)^3} \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(S_1\,r_1, S_2\,r_2, S_3\,r_3)\})\,\mathrm{d}\,\mathbb{P}^{(R_1,R_2,R_3)}((r_1,r_2,r_3))\\
&= 2^3 \int_{(\mathbb{R}\times\mathbb{R}_+)^3} \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(S_1\,r_1, S_2\,r_2, S_3\,r_3)\})\,\mathrm{d}\,\mathbb{P}^{(R_1,R_2,R_3)}((r_1,r_2,r_3))\\
&= 2^3 \sum_{\pi\in\Pi(1,2,3)} \int_{\mathbb{A}_3(\pi)} \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(S_1\,r_1, S_2\,r_2, S_3\,r_3)\})\,\mathrm{d}\,\mathbb{P}^{(R_1,R_2,R_3)}((r_1,r_2,r_3))\\
&= 2^3 \sum_{\pi\in\Pi(1,2,3)} \int_{\mathbb{A}_3(\pi)} \frac{1}{4}\,\mathrm{d}\,\mathbb{P}^{(R_1,R_2,R_3)}((r_1,r_2,r_3))\\
&= \frac{1}{4}\,2^3 \sum_{\pi\in\Pi(1,2,3)} \mathbb{P}^{(R_1,R_2,R_3)}(\mathbb{A}_3(\pi)) = \frac{1}{4}
\end{aligned}$$

since $\mathbb{P}^{(R_1,R_2,R_3)}(\mathbb{A}_3(\pi)) = \frac{1}{3!}\frac{1}{2^3} = \frac{1}{6}\frac{1}{2^3}$ according to (S.3). □

As soon as there are four residuals r_1, r_2, r_3, r_4, there exist situations with $0_2 \notin \mathbb{S}(s_1\, r_1, s_2\, r_2, s_3\, r_3) \cap \mathbb{S}(s_2\, r_2, s_3\, r_3, s_4\, r_4)$ for all $(s_1, s_2, s_3, s_4) \in \{-1, 1\}^2$. The reason is that s_2 and s_3 must appear simultaneously in $\mathbb{S}(s_1\, r_1, s_2\, r_2, s_3\, r_3)$ as well as in $\mathbb{S}(s_2\, r_2, s_3\, r_3, s_4\, r_4)$.

Let π^{-1} denote the inverse mapping of the permutation π. If π leads to the ordering

$$\begin{aligned}\arccos(r_{\pi(1),1}/\|r_{\pi(1)}\|) < \arccos(r_{\pi(2),1}/\|r_{\pi(2)}\|) < \arccos(r_{\pi(3),1}/\|r_{\pi(3)}\|)\\ < \arccos(r_{\pi(4),1}/\|r_{\pi(4)}\|),\end{aligned}$$

then $\pi^{-1}(i)$ provides the position of r_i in this ordering for $i = 1, \ldots, 4$. For example, for $\pi = (2, 4, 1, 3)$ we get $\arccos(r_{2,1}/\|r_2\|) < \arccos(r_{4,1}/\|r_4\|) < \arccos(r_{1,1}/\|r_1\|) < \arccos(r_{3,1}/\|r_3\|)$ and $\pi^{-1}(2) = 1$, $\pi^{-1}(3) = 4$, i.e., r_2 is at the first position and r_3 is at the last position. Note also $\pi = (2, 4, 1, 3) \in E_4((1, 3, 2, 4))$.

Lemma S.3 *If* $(r_1, r_2, r_3, r_4) \in \mathbb{A}_4(\pi)$ *with* $\pi = (\pi(1), \pi(2), \pi(3), \pi(4)) \in \Pi(1, 2, 3, 4)$*, then the following assertions are equivalent:*

(a) Exactly two $(s_1, s_2, s_3, s_4) \in \{-1, 1\}^4$ *exist with*
$$0_2 \in \mathbb{S}(s_1\, r_1, s_2\, r_2, s_3\, r_3) \cap \mathbb{S}(s_2\, r_2, s_3\, r_3, s_4\, r_4).$$

(b) $|\pi^{-1}(2) - \pi^{-1}(3)| = 1$ *or* $\{\pi^{-1}(2), \pi^{-1}(3)\} \in \{1, 4\}$.

(c) $\pi \in E_4((1, 2, 3, 4)) \cup E_4((4, 2, 3, 1)) \cup E_4((4, 3, 2, 1)) \cup E_4((1, 3, 2, 4))$.

The characterization (b) in Lemma S.3 was already given in Dyckerhoff et al. (2015).

Proof. Since $(r_1, r_2, r_3, r_4) \in \mathbb{A}_4(\pi) \subset (\mathbb{R}\times\mathbb{R}_+)^4$, we have $(r_1, r_2, r_3) \in \mathbb{A}_3(\pi_1)$ and $(r_2, r_3, r_4) \in \mathbb{A}_3(\pi_2)$ with $\pi_1, \pi_2 \in \Pi(1, 2, 3)$. Lemma S.2 provides

i) $0_2 \in \mathbb{S}(s_1^1\, r_1, s_2^1\, r_2, s_3^1\, r_3)$ and $s_1^1 = 1$ if and only if $(s^1_{\pi_1(1)}, s^1_{\pi_1(2)}, s^1_{\pi_1(3)}) = (1, -1, 1)$ or $(s^1_{\pi_1(1)}, s^1_{\pi_1(2)}, s^1_{\pi_1(3)}) = (-1, 1, -1)$,

ii) $0_2 \in \mathbb{S}(s_1^2\, r_2, s_2^2\, r_3, s_3^2\, r_4)$ and $s_1^2 = 1$ if and only if $(s^2_{\pi_2(1)}, s^2_{\pi_2(2)}, s^2_{\pi_2(3)}) = (1, -1, 1)$ or $(s^2_{\pi_2(1)}, s^2_{\pi_2(2)}, s^2_{\pi_2(3)}) = (-1, 1, -1)$.

Case i) means either $s_2^1 = -s_3^1$ or $s_2^1 = s_3^1$. If $s_2^1 = -s_3^1$ holds, then r_2 and r_3 are neighbours in the ordering of π_1. If $s_2^1 = s_3^1$ holds, then r_1 is lying between r_2 and r_3 in the ordering of π_1. Similarly, Case ii) means either $s_1^2 = -s_2^2$ or $s_1^2 = s_2^2$. If $s_1^2 = -s_2^2$ holds, then r_2 and r_3 are neighbours in the ordering of π_2. If $s_1^1 = s_2^1$ holds, then r_4 is lying between r_2 and r_3 in the ordering of π_2. Hence if $(s_1, s_2, s_3, s_4) \in \{-1, 1\}^4$ exists with $0_2 \in \mathbb{S}(s_1\, r_1, s_2\, r_2, s_3\, r_3) \cap \mathbb{S}(s_2\, r_2, s_3\, r_3, s_4\, r_4)$, then $(s_2, s_3) = \pm(s_2^1, s_3^1) = \pm(s_1^2, s_2^2)$. This means that either r_2 and r_3 are neighbours in the ordering π, i.e. $2, 3$ are neighbours in π, or r_1 and r_4 are lying between r_2 and r_3 in the ordering of π_2, i.e. $1, 4$ are lying between $2, 3$ in π. However, this is the assertion of (b). Assertion (c) is only presenting the possible cases of (b) more explicitly. Note also, that the case that $1,4$ are lying between $2,3$ in π also means that $2,3$ are neighbours in a cyclic sense. □

Alternative proof of Theorem 2 (b). Define

$$\Pi_4^0 := E_4((1,2,3,4)) \cup E_4((4,2,3,1)) \cup E_4((4,3,2,1)) \cup E_4((1,3,2,4)).$$

According to Lemma S.3, the set Π_4^0 is exactly the set of all permutations π so that for any $(r_1,r_2,r_3,r_4) \in \mathbb{A}_4(\pi)$ a $(s_1,s_2,s_3,s_4) \in \{-1,1\}^4$ exists with $0_2 \in \mathbb{S}(s_1\,r_1, s_2\,r_2, s_3\,r_3) \cap \mathbb{S}(s_2\,r_2, s_3\,r_3, s_4\,r_4)$. This set has 4·4 elements since it consist of four equivalence classes and each equivalence class has four elements. Note also that the equivalence classes $E_4((1,2,3,4))$ and $E_4((4,2,3,1))$ are related by interchanging the positions of 1 and 4. The same relation holds for the equivalence classes $E_4((4,3,2,1))$ and $E_4((1,3,2,4))$. The equivalence classes $E_4((1,2,3,4))$ and $E_4((1,3,2,4)$ are related by interchanging the order of 2 and 3, and the same holds for $E_4((4,2,3,1))$ and $E_4((4,3,2,1))$. Hence we also can express the number of elements in Π_4^0 as $\sharp(\Pi_4^0) = 4\cdot 2\cdot 2$. Property (S.3) yields here

$$\mathbb{P}^{(R_1,R_2,R_3,R_4)}(\mathbb{A}_4(\pi)) = \frac{1}{2^4}\frac{1}{4!}.$$

Moreover, Lemma S.3 provides for any $(r_1,r_2,r_3,r_4) \in A_4(\pi)$ with $\pi \in \Pi_4^0$ that exactly two vectors $(s_1^1,s_2^1,s_3^1,s_4^1)$ and $(s_1^2,s_2^2,s_3^2,s_4^2) = -(s_1^1,s_2^1,s_3^1,s_4^1)$ exist so that $0_2 \in \mathbb{S}(s_1^i\,r_1, s_2^i\,r_2, s_3^i\,r_3) \cap \mathbb{S}(s_2^i\,r_2, s_3^i\,r_3, s_4^i\,r_4)$ for $i=1,2$. Hence

$$\begin{aligned}
&\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(S_1\,r_1, S_2\,r_2, S_3\,r_3) \cap \mathbb{S}(S_2\,r_2, S_3\,r_3, S_4\,r_4)\}) \\
&= \mathbb{P}^{(S_1,S_2,S_3,S_4)}(\{(s_1^1,s_2^1,s_3^1,s_4^1),(s_1^2,s_2^2,s_3^2,s_4^2)\}) = \frac{2}{2^4} = \frac{1}{8}.
\end{aligned}$$

So, as in the alternative proof of Theorem 2 (a), we get

$$\begin{aligned}
&\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1,R_2,R_3) \cap \mathbb{S}(R_2,R_3,R_4)\}) \\
&= \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(S_1\,R_1, S_2\,R_2, S_3\,R_3) \cap \mathbb{S}(S_2\,R_2, S_3\,R_3, S_4\,R_4)\}) \\
&= 2^4 \int_{(\mathbb{R}\times\mathbb{R}_+)^4} \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(S_1\,r_1, S_2\,r_2, S_3\,r_3) \cap \mathbb{S}(S_2\,r_2, S_3\,r_3, S_4\,r_4)\}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\,\mathbb{P}^{(R_1,R_2,R_3,R_4)}((r_1,r_2,r_3,r_4)) \\
&= 2^4 \sum_{\pi\in\Pi_4^0} \int_{\mathbb{A}_4(\pi)} \frac{2}{2^4}\,\mathrm{d}\,\mathbb{P}^{(R_1,R_2,R_3,R_4)}((r_1,r_2,r_3,r_4)) \\
&= \frac{2}{2^4}\, 2^4 \sum_{\pi\in\Pi_4^0} \mathbb{P}^{(R_1,R_2,R_3,R_4)}(\mathbb{A}_4(\pi)) \\
&= \frac{2}{2^4}\, 2^4 \cdot 4\cdot 2\cdot 2 \cdot \frac{1}{2^4}\frac{1}{4!} = \frac{2}{2^4}\,\frac{4\cdot 2\cdot 2}{4!} = \frac{1}{12}
\end{aligned}$$

as we wanted to show. □

Alternative proof of Theorem 2 (c). As in the proof of Theorem 2 (b), the main step is to determine the cardinality of Π_5^0, which is the set of all $\pi \in \Pi(1,2,\ldots,5)$ so that for

$$\begin{aligned}&(r_1, r_2, r_3, r_4, r_5) \in \mathbb{A}_5(\pi) \text{ two vectors } (s_1, s_2, s_3, s_4, s_5) \in \{-1,1\}^5\\ &\text{exist with} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (S.4)\\ &0_2 \in \mathbb{S}(s_1\, r_1, s_2\, r_2, s_3\, r_3) \cap \mathbb{S}(s_2\, r_2, s_3\, r_3, s_4\, r_4) \cap \mathbb{S}(s_3\, r_3, s_4\, r_4, s_5\, r_5).\end{aligned}$$

This problem reduces to the problem of determining the number of equivalence classes $E_5(\pi_{rp}) \subset \Pi(1,2,\ldots,5)$ with representatives π_{rp} satisfying (S.4). This means that $(r_1, r_2, r_3, r_4, r_5) \in \mathbb{A}_5(\pi_{rp})$ simultaneously satisfies

$$0_2 \in \mathbb{S}(s_1\, r_1, s_2\, r_2, s_3\, r_3) \cap \mathbb{S}(s_2\, r_2, s_3\, r_3, s_4\, r_4)$$

and

$$0_2 \in \mathbb{S}(s_2\, r_2, s_3\, r_3, s_4\, r_4) \cap \mathbb{S}(s_3\, r_3, s_4\, r_4, s_5\, r_5)$$

for some $(s_1, s_2, s_3, s_4, s_5) \in \{-1,1\}^5$. According to Lemma S.3, the components of π_{rp} which concern $1,2,3,4$ must have an order as in

$$E_4((1,2,3,4)) \cup E_4((4,2,3,1)) \cup E_4((4,3,2,1)) \cup E_4((1,3,2,4)). \quad (S.5)$$

Analogously, the components of π_{rp} which concern $2,3,4,5$ must have an order as in

$$E_4((2,3,4,5)) \cup E_4((5,3,4,2)) \cup E_4((5,4,3,2)) \cup E_4((2,4,3,5)). \quad (S.6)$$

The representatives of the equivalence classes in (S.6) are obtained by adding 1 to the values of the representatives of the equivalence classes in (S.5). In all vectors in the equivalence classes in (S.5), 2 and 3 are direct neighbours (possibly in the cyclic sense), which means

$$\begin{aligned}&\text{neither of 1 and 4 is lying between 2 and 3}\\ &\qquad\text{or both of 1 and 4 are lying between 2 and 3.}\end{aligned} \quad (S.7)$$

In all vectors in the equivalence classes in (S.6), 3 and 4 are direct neighbours (again possibly in the cyclic sense), which means

$$\begin{aligned}&\text{neither of 2 and 5 is lying between 3 and 4}\\ &\qquad\text{or both of 2 and 5 are lying between 3 and 4.}\end{aligned} \quad (S.8)$$

Hence for 2,3,4 we have only the following two possibilities of ordering:

$$\text{“}2,3,4\text{”},\ \text{“}4,3,2\text{”} \quad (S.9)$$

where “2,3,4” means “2 before 3 and 3 before 4” and “4,3,2” means “4 before 3 and 3 before 2”. If we fix the order as “2,3,4”, then we have the following possibilities for the location of 1 and 5 retaining the ordering “2,3,4” and

satisfying (S.7) and (S.8):
For 1:

$$1 \text{ appears before 2, or equivalently after 4,} \tag{S.10}$$
$$1 \text{ appears between 3 and 4.} \tag{S.11}$$

For 5:

$$5 \text{ appears before 2, or equivalently after 4,} \tag{S.12}$$
$$5 \text{ appears between 2 and 3.} \tag{S.13}$$

The possibilities (S.10) and (S.11) can be combined arbitrarily with the possibilities (S.12) and (S.13), so that there are 4 combinations. In the combination (S.10)–(S.12), both 1 and 5 appear before 2 or equivalently after 4. Here we have two possible orders: "1 before 5" and "5 before 1". Hence, there are $2 \cdot 2 + 1 = 5$ locations of 1 and 5 retaining the ordering "2,3,4" and satisfying (S.7) and (S.8). The same number of possible locations of 1 and 5 holds also for the other order "4,3,2" in (S.9). This leads to $2 \cdot 5$ representatives $\pi_{rp} \in \Pi(1, 2, \ldots, 5)$ and thus $2 \cdot 5$ equivalent classes $E_5(\pi_{rp})$ with representatives π_{rp} satisfying (S.4). Since every equivalence class $E_5(\pi_{rp})$ has 5 elements, we get

$$\sharp(\Pi_5^0) = 5 \cdot 2 \cdot 5 = 50.$$

Property (S.3) yields here

$$\mathbb{P}^{(R_1,R_2,R_3,R_4,R_5)}(\mathbb{A}_5(\pi)) = \frac{1}{2^5}\frac{1}{5!}$$

for any $\pi \in \Pi(1, 2, \ldots, 5)$. The rest of the proof follows as in the proof of Theorem 2 (b): Lemma S.3 provides for any $(r_1, r_2, r_3, r_4, r_5) \in A_5(\pi)$ with $\pi \in \Pi_5^0$ that exactly to vectors $(s_1^1, s_2^1, s_3^1, s_4^1, s_5^1)$ and $(s_1^2, s_2^2, s_3^2, s_4^2, s_4^2) = -(s_1^1, s_2^1, s_3^1, s_4^1, s_5^1)$ exist so that

$$0_2 \in \mathbb{S}(s_1^i\, r_1, s_2^i\, r_2, s_3^i\, r_3) \cap \mathbb{S}(s_2^i\, r_2, s_3^i\, r_3, s_4^i\, r_4) \cap \mathbb{S}(s_3^i\, r_3, s_4^i\, r_4, s_5^i\, r_5)$$

for $i = 1, 2$. Hence

$$\begin{aligned}
&\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(S_1\, r_1, S_2\, r_2, S_3\, r_3) \cap \mathbb{S}(S_2\, r_2, S_3\, r_3, S_4\, r_4) \cap \mathbb{S}(S_3\, r_3, S_4\, r_4, S_5\, r_5)\}) \\
&\quad = \mathbb{P}^{(S_1,S_2,S_3,S_4,S_5)}(\{(s_1^1, s_2^1, s_3^1, s_4^1, s_5^1), (s_1^2, s_2^2, s_3^2, s_4^2, s_5^2)\}) = \frac{2}{2^5} = \frac{1}{2^4},
\end{aligned}$$

so that

$$\begin{aligned}
&\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1, R_2, R_3) \cap \mathbb{S}(R_2, R_3, R_4) \cap \mathbb{S}(R_3, R_4, R_5)\}) \\
&= 2^5 \sum_{\pi \in \Pi_5^0} \int_{\mathbb{A}_5(\pi)} \frac{2}{2^5}\, \mathrm{d}\, \mathbb{P}^{(R_1,R_2,R_3,R_4,R_5)}((r_1, r_2, r_3, r_4, r_5)) \\
&= \frac{2}{2^5}\, 2^5 \sum_{\pi \in \Pi_5^0} \mathbb{P}^{(R_1,R_2,R_3,R_4,R_5)}(\mathbb{A}_5(\pi))
\end{aligned}$$

$$= \frac{2}{2^5}\, 2^5 \cdot 5 \cdot 2 \cdot 5 \cdot \frac{1}{2^5}\frac{1}{5!} = \frac{2}{2^5}\,\frac{5 \cdot 2 \cdot 5}{5!} = \frac{1}{2^4}\,\frac{5}{12}.$$

The proof is finished. □

Alternative proof of Theorem 2 (d). As in the proofs of Theorem 2 (b) and (c), the main step is to determine the cardinality of Π_6^0, which is the set of all $\pi \in \Pi(1,2,\dots,6)$ so that for

$$\begin{aligned}&(r_1,r_2,r_3,r_4,r_5,r_6) \in \mathbb{A}_6(\pi) \text{ two vectors } (s_1,s_2,s_3,s_4,s_5,s_6) \in \{-1,1\}^6\\ &\text{ exist with} \qquad\qquad (\text{S.14})\\ &0_2 \in \mathbb{S}(s_1\,r_1, s_2\,r_2, s_3\,r_3) \cap \mathbb{S}(s_2\,r_2, s_3\,r_3, s_4\,r_4)\\ &\qquad \cap \mathbb{S}(s_3\,r_3, s_4\,r_4, s_5\,r_5) \cap \mathbb{S}(s_4\,r_4, s_5\,r_5, s_6\,r_6).\end{aligned}$$

Again, we have to determine the number of equivalence classes $E_6(\pi_{rp}) \subset \Pi(1,2,\dots,6)$ with representatives π_{rp} satisfying (S.14). This means that $(r_1,r_2,r_3,r_4,r_5,r_6) \in \mathbb{A}_6(\pi_{rp})$ simultaneously satisfies

$$\begin{aligned}0_2 &\in \mathbb{S}(s_1\,r_1, s_2\,r_2, s_3\,r_3) \cap \mathbb{S}(s_2\,r_2, s_3\,r_3, s_4\,r_4),\\ 0_2 &\in \mathbb{S}(s_2\,r_2, s_3\,r_3, s_4\,r_4) \cap \mathbb{S}(s_3\,r_3, s_4\,r_4, s_5\,r_5),\\ 0_2 &\in \mathbb{S}(s_3\,r_3, s_4\,r_4, s_5\,r_5) \cap \mathbb{S}(s_4\,r_4, s_5\,r_5, s_6\,r_6),\end{aligned}$$

for some $(s_1,s_2,s_3,s_4,s_5,s_6) \in \{-1,1\}^6$. Analogously, as in the proof of Theorem 2 (c), this implies that

$$\begin{aligned}&\text{neither of 1 and 4 is lying between 2 and 3}\\ &\qquad \text{or both of 1 and 4 are lying between 2 and 3,} \qquad (\text{S.15})\\ &\text{neither of 2 and 5 is lying between 3 and 4}\\ &\qquad \text{or both of 2 and 5 are lying between 3 and 4,} \qquad (\text{S.16})\\ &\text{neither of 3 and 6 is lying between 4 and 5}\\ &\qquad \text{or both of 3 and 6 are lying between 4 and 5.} \qquad (\text{S.17})\end{aligned}$$

Hence for 2,3,4,5 we have only the following possibilities of ordering:

$$\text{“}2,3,4,5\text{”},\ \text{“}5,4,3,2\text{”}, \qquad (\text{S.18})$$

$$\text{“}2,5,3,4\text{”},\ \text{“}4,3,5,2\text{”}. \qquad (\text{S.19})$$

Note that the first orderings in (S.18) as well as in (S.19) are the reverse orderings of the second ones in (S.18) and (S.19), respectively. Moreover, the ordering “2,5,3,4” of the first ordering in (S.19) will lead to the same equivalence class as “3,4,2,5”.

If we fix the order as “2,3,4,5”, then we have the following possibilities for the location of 1 and 6 retaining the ordering “2,3,4,5” and satisfying (S.15), (S.16), (S.17):
For 1:

$$\text{1 appears before 2, or equivalently after 5,} \qquad (\text{S.20})$$

$$1 \text{ appears between 3 and 4,} \tag{S.21}$$

$$1 \text{ appears between 4 and 5.} \tag{S.22}$$

For 6:

$$6 \text{ appears before 2, or equivalently after 5,} \tag{S.23}$$

$$6 \text{ appears between 2 and 3,} \tag{S.24}$$

$$6 \text{ appears between 3 and 4.} \tag{S.25}$$

The possibilities (S.20), (S.21), and (S.22) can be combined arbitrarily with the possibilities (S.23), (S.24), and (S.25), so that there are $3 \cdot 3$ combinations. In the combination (S.20)–(S.23), both 1 and 6 appear before 2 or equivalently after 5. Additionally, in the combination (S.21)–(S.25), both 1 and 6 appear between 3 and 4. In both cases, we have two possible orders: "1 before 6" and "6 before 1" so that we have to add 2 to $3{\cdot}3$. Hence there are $3{\cdot}3{+}2 = 11$ locations of 1 and 6 retaining the ordering "2,3,4,5" and satisfying (S.15), (S.16), (S.17). The same number of possible locations of 1 and 6 also holds for the other order "5,4,3,2" in (S.18). This leads to $2 \cdot 11$ combinations for the orderings in (S.18).

If we fix the order as "2,5,3,4", then we have the following possibilities for the location of 1 and 6 retaining the ordering "2,5,3,4" and satisfying (S.15), (S.16), (S.17):
For 1:

$$1 \text{ appears before 2, or equivalently after 4,} \tag{S.26}$$

$$1 \text{ appears between 3 and 4.} \tag{S.27}$$

For 6:

$$6 \text{ appears between 5 and 3,} \tag{S.28}$$

$$6 \text{ appears between 3 and 4.} \tag{S.29}$$

The possibilities (S.26) and (S.27) can be combined arbitrarily with the possibilities (S.28) and (S.29), so that there are $2 \cdot 2$ combinations. In the combination (S.27)–(S.29), both 1 and 6 appear between 3 and 4. Hence, there are $2 \cdot 2 + 1 = 5$ locations of 1 and 6 retaining the ordering "2,5,3,4" and satisfying (S.15), (S.16), (S.17). The same number of possible locations of 1 and 6 holds also for the other order "4,3,5,2" in (S.19). This leads to $2 \cdot 5$ combinations for the orderings in (S.19).

All together, this leads to $2 \cdot 11 + 2 \cdot 5 = 2 \cdot 16 = 32$ representatives $\pi_{rp} \in \Pi(1, 2, \ldots, 6)$ and thus 32 equivalent classes $E_6(\pi_{rp})$ with representatives π_{rp} satisfying (S.14). Since every equivalence class $E_6(\pi_{rp})$ has 6 elements, we get

$$\sharp(\Pi_6^0) = 6 \cdot (2 \cdot 11 + 2 \cdot 5) = 6 \cdot 32 = 192.$$

As in the proof of Theorem 2 (c), we obtain

$$\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1, R_2, R_3) \cap \mathbb{S}(R_2, R_3, R_4) \cap \mathbb{S}(R_3, R_4, R_5) \cap \mathbb{S}(R_4, R_5, R_6)\})$$

$$= 2^6 \sum_{\pi \in \Pi_6^0} \int_{\mathbb{A}_6(\pi)} \frac{2}{2^6} \, \mathrm{d}\, \mathbb{P}^{(R_1,R_2,R_3,R_4,R_5,R_6)}((r_1,r_2,r_3,r_4,r_5,r_6))$$
$$= \frac{2}{2^6}\, 2^6 \sum_{\pi \in \Pi_6^0} \mathbb{P}^{(R_1,R_2,R_3,R_4,R_5,R_6)}(\mathbb{A}_6(\pi))$$
$$= \frac{2}{2^6}\, 2^6 \cdot 6 \cdot (2 \cdot 11 + 2 \cdot 5) \cdot \frac{1}{2^6} \frac{1}{6!} = \frac{1}{2^5} \frac{6 \cdot 32}{6!} = \frac{1}{2^5} \frac{4}{15}.$$

□

With the methods used above, a special case of Theorem 2 (f) can be proved as well. This special case is given in the follwoing lemma.

Lemma S.4 *The random variables*

$\mathbb{1}\{0_2 \in \mathbb{S}(R_1,R_2,R_3) \cap \mathbb{S}(R_2,R_3,R_4)\}$ *and* $\mathbb{1}\{0_2 \in \mathbb{S}(R_4,R_5,R_6) \cap \mathbb{S}(R_5,R_6,R_7)\}$

are stochastically independent.

Proof. Since the random variables are indicator functions, we have only to show

$$\begin{aligned}
&\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1,R_2,R_3) \cap \mathbb{S}(R_2,R_3,R_4)\} \\
&\qquad\qquad \cdot \mathbb{1}\{0_2 \in \mathbb{S}(R_4,R_5,R_6) \cap \mathbb{S}(R_5,R_6,R_7)\}) \\
&= \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1,R_2,R_3) \cap \mathbb{S}(R_2,R_3,R_4)\}) \\
&\qquad\qquad \cdot \mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_4,R_5,R_6) \cap \mathbb{S}(R_5,R_6,R_7)\}) \\
&= \frac{1}{12} \cdot \frac{1}{12}
\end{aligned}$$

where the last equality follows from Theorem 2 (b). As in the first proof of Theorem 2 (b), the main step is to determine the cardinality of Π_7^0, which is the set of all $\pi \in \Pi(1,2,\ldots,7)$ so that for

$$\begin{aligned}
&(r_1,r_2,\ldots,r_7) \in \mathbb{A}_7(\pi) \text{ two vectors } (s_1,s_2,\ldots,s_7) \in \{-1,1\}^7 \\
&\text{ exists with} \\
&0_2 \in \mathbb{S}(s_1\, r_1, s_2\, r_2, s_3\, r_3) \,\cap\, \mathbb{S}(s_2\, r_2, s_3\, r_3, s_4\, r_4) \\
&\qquad\qquad \text{and } 0_2 \in \mathbb{S}(s_4\, r_4, s_5\, r_5, s_6\, r_6) \,\cap\, \mathbb{S}(s_5\, r_5, s_6\, r_6, s_7\, r_7).
\end{aligned} \tag{S.30}$$

This problem reduces to the problem of determining the number of equivalence classes $E_7(\pi_{rp}) \subset \Pi(1,2,\ldots,7)$ with representatives π_{rp} satisfying (S.30). This means that $(r_1,r_2,\ldots,r_7) \in \mathbb{A}_7(\pi_{rp})$ simultaneously satisfies

$$0_2 \in \mathbb{S}(s_1\, r_1, s_2\, r_2, s_3\, r_3) \,\cap\, \mathbb{S}(s_2\, r_2, s_3\, r_3, s_4\, r_4)$$

and

$$0_2 \in \mathbb{S}(s_4\, r_4, s_5\, r_5, s_6\, r_6) \,\cap\, \mathbb{S}(s_5\, r_5, s_6\, r_6, s_7\, r_7)$$

for some $(s_1, s_2, \ldots, s_7) \in \{-1, 1\}^7$. According to Lemma S.3, the components of π_{rp} which concern $1, 2, 3, 4$ must have an order as in

$$E_4((1,2,3,4)) \cup E_4((4,2,3,1)) \cup E_4((4,3,2,1)) \cup E_4((1,3,2,4)). \quad \text{(S.31)}$$

Analogously, the components of π_{rp} which concern $4, 5, 6, 7$ must have an order as in

$$E_4((4,5,6,7)) \cup E_4((7,5,6,4)) \cup E_4((7,6,5,4)) \cup E_4((4,6,5,7)). \quad \text{(S.32)}$$

The representatives of the equivalence classes in (S.32) are obtained by adding 3 to the values of the representatives of the equivalence classes in (S.31). Since the component 4 is the shared component in (S.31) and (S.32), all permutations in (S.31) can be combined with all combinations in (S.32). However, they provide different merging possibilities. While $(1, 2, 3, 4)$ and $(4, 6, 5, 7)$ only can be merged to $(1, 2, 3, 4, 6, 5, 7)$, there are the following merging possibilities, for example, of $(3, 2, 4, 1) \in E_4((1, 3, 2, 4))$ and $(6, 4, 5, 7) \in E_4((7, 5, 6, 4))$:

$$\begin{aligned}
&(6,3,2,4,1,5,7), (3,6,2,4,1,5,7), (3,2,6,4,1,5,7), \\
&(6,3,2,4,5,1,7), (3,6,2,4,5,1,7), (3,2,6,4,5,1,7), \\
&(6,3,2,4,5,7,1), (3,6,2,4,5,7,1), (3,2,6,4,5,7,1).
\end{aligned} \quad \text{(S.33)}$$

In this example, component 4 is at the third position of the element $(3, 2, 4, 1)$ of class (S.31) and at the second position of the element $(6, 4, 5, 7)$ of (S.32) so that the components 3,2,6 should be at the 3 positions before 4 and the components 1,5,7 at the 3 positions after 4. There are $\binom{3}{2}$ positions for 3,2 for coming with 6 before 4 and $\binom{3}{1}$ positions for 1 for coming with 5,7 after 4. This provides the $3 \cdot 3$ permutations given in (S.33).

In each of the equivalence classes in (S.31) and (S.32), the component 4 can be at first, second, third or fourth position. Hence there are $4 \cdot 4$ different merging situations, which are given in Table S.1.

Since each of the equivalence classes has 4 elements, the number of all possible permutations is given by

$$\sharp(\Pi_7^0) = (4 \cdot 4) \cdot (4 \cdot 5 \cdot 7)$$

so that, as in the proofs of Theorem 2 (c) and (d), we obtain

$$\begin{aligned}
&\mathbb{E}(\mathbb{1}\{0_2 \in \mathbb{S}(R_1, R_2, R_3) \cap \mathbb{S}(R_2, R_3, R_4)\} \\
&\qquad\qquad \cdot \mathbb{1}\{0_2 \in \mathbb{S}(R_4, R_5, R_6) \cap \mathbb{S}(R_5, R_6, R_7)\}) \\
&= \frac{2}{2^7}\, 2^7\, (4 \cdot 4 \cdot 4 \cdot 5 \cdot 7) \cdot \frac{1}{2^7} \frac{1}{7!} = \frac{1}{2^6} \frac{4 \cdot 4 \cdot 4}{4!\, 6} = \frac{1}{2^6} \frac{4 \cdot 4}{6 \cdot 6} = \frac{1}{12} \cdot \frac{1}{12}.
\end{aligned}$$

□

Alternative proof of Lemma S.4. The number of possible permutations $\sharp(\Pi_7^0)$ can also be obtained as in the proofs of Theorem 2 (c) and (d). However, this proof is lengthier than the above proof.

Table S.1 Merging situations of the equivalence classes in (S.31) and (S.32).

Position of 4 in class (S.31)	class (S.32)	Number of mergings	
1	1	$\binom{6}{3} =$	20
1	2	$\binom{5}{2} =$	10
1	3	$\binom{4}{1} =$	4
1	4	$1 =$	1
2	1	$\binom{1}{1} \cdot \binom{5}{2} = 1 \cdot 10 =$	10
2	2	$\binom{2}{1} \cdot \binom{4}{2} = 2 \cdot 6 =$	12
2	3	$\binom{3}{1} \cdot \binom{3}{2} = 3 \cdot 3 =$	9
2	4	$\binom{4}{1} \cdot \binom{2}{2} = 4 \cdot 1 =$	4
3	1	$\binom{2}{2} \cdot \binom{4}{1} = 1 \cdot 4 =$	4
3	2	$\binom{3}{2} \cdot \binom{3}{1} = 3 \cdot 3 =$	9
3	3	$\binom{4}{2} \cdot \binom{2}{1} = 6 \cdot 2 =$	12
3	4	$\binom{5}{2} \cdot \binom{1}{1} = 10 \cdot 1 =$	10
4	1	$1 =$	1
4	2	$\binom{4}{1} =$	4
4	3	$\binom{5}{2} =$	10
4	4	$\binom{6}{3} =$	20
Sum			$4 \cdot 35 = 4 \cdot 5 \cdot 7$

To determine the number of equivalence classes $E_7(\pi_{rp}) \subset \Pi(1, 2, \ldots, 7)$ with representatives π_{rp} satisfying (S.30) means that any $(r_1, r_2, \ldots, r_7) \in \mathbb{A}_7(\pi_{rp})$ simultaneously satisfies

$$0_2 \in \mathbb{S}(s_1\, r_1, s_2\, r_2, s_3\, r_3) \cap \mathbb{S}(s_2\, r_2, s_3\, r_3, s_4\, r_4)$$

and

$$0_2 \in \mathbb{S}(s_4\, r_4, s_5\, r_5, s_6\, r_6) \cap \mathbb{S}(s_5\, r_5, s_6\, r_6, s_7\, r_7)$$

for some $(s_1, s_2, \ldots, s_7) \in \{-1, 1\}^7$. Analogously, as in the proof of Theorem 2 (c) and (d), this implies that

$$\begin{aligned} &\text{neither of 1 and 4 is lying between 2 and 3} \\ &\qquad \text{or both of 1 and 4 are lying between 2 and 3,} \qquad \text{(S.34)} \\ &\text{neither of 4 and 7 is lying between 5 and 6} \\ &\qquad \text{or both of 4 and 7 are lying between 5 and 6.} \qquad \text{(S.35)} \end{aligned}$$

To determine all representatives π_{rp} satisfying (S.30), we consider the following equivalence classes $E_7(\pi_{rp})$

Class “2,3,5,6” : 2 before 3, 5 before 6 : 4 is not lying between 2 and 3,

$$\begin{aligned}
& & & \text{4 is not lying between 5 and 6.}\\
&\textbf{Class “3,2,5,6”}: && 3\,\text{before}\,2,\ 5\,\text{before}\,6: \text{4 is not lying between 3 and 2,}\\
& & & \text{4 is not lying between 5 and 6.}\\
&\textbf{Class “2,3,6,5”}: && 2\,\text{before}\,3,\ 6\,\text{before}\,5: \text{4 is not lying between 2 and 3,}\\
& & & \text{4 is not lying between 6 and 5.}\\
&\textbf{Class “3,2,6,5”}: && 3\,\text{before}\,2,\ 6\,\text{before}\,5: \text{4 is not lying between 3 and 2,}\\
& & & \text{4 is not lying between 6 and 5.}
\end{aligned}$$

We consider first **Class “2,3,5,6”**. This class has the following 6 subclasses given by

$$\text{“}2,3,4,5,6\text{”};\ \text{“}4,2,3,5,6\text{”};\ \text{“}4,2,5,3,6\text{”};$$
$$\text{“}4,5,2,6,3\text{”};\ \text{“}4,2,5,6,3\text{”};\ \text{“}4,5,2,3,6\text{”}.$$

For each of these subclasses, we determine now how many positions of 1 and 7 in a representative π_{rp} are possible so that (S.34) and (S.35) are satisfied.

1. Class “2,3,4,5,6”
For 1:

$$\text{1 appears before 2, equivalently after 6,} \tag{S.36}$$
$$\text{1 appears between 3 and 4,} \tag{S.37}$$
$$\text{1 appears between 4 and 5,} \tag{S.38}$$
$$\text{1 appears between 5 and 6.} \tag{S.39}$$

For 7:

$$\text{7 appears before 2, equivalently after 6,} \tag{S.40}$$
$$\text{7 appears between 2 and 3,} \tag{S.41}$$
$$\text{7 appears between 3 and 4,} \tag{S.42}$$
$$\text{7 appears between 4 and 5.} \tag{S.43}$$

Since two orders of 1 and 7 are possible in the combinations (S.36)–(S.40), (S.37)–(S.42), (S.38)–(S.43), this leads to $4 \cdot 4 + 3 = 19$ combinations and thus 19 different representatives π_{rp}.

2. Class “4,2,3,5,6”
For 1:

$$\text{1 appears before 4, equivalently after 6,} \tag{S.44}$$
$$\text{1 appears between 4 and 2,} \tag{S.45}$$
$$\text{1 appears between 3 and 5,} \tag{S.46}$$
$$\text{1 appears between 5 and 6.} \tag{S.47}$$

For 7:

$$\text{7 appears before 4, equivalently after 6,} \tag{S.48}$$

7 appears between 4 and 2, (S.49)

7 appears between 2 and 3, (S.50)

7 appears between 3 and 5. (S.51)

Since two orders of 1 and 7 are possible in the combinations (S.44)–(S.48), (S.45)–(S.49), (S.46)–(S.51), this leads to $4 \cdot 4 + 3 = 19$ combinations and thus 19 different representatives π_{rp}.

3. Class "4,2,5,3,6"
For 1:

1 appears before 4, equivalently after 6, (S.52)

1 appears between 4 and 2, (S.53)

1 appears between 3 and 6. (S.54)

For 7:

7 appears before 4, equivalently after 6, (S.55)

7 appears between 4 and 2, (S.56)

7 appears between 2 and 5. (S.57)

Since two orders of 1 and 7 are possible in the combinations (S.52)–(S.55), (S.53)–(S.56), this leads to $3 \cdot 3 + 2 = 11$ combinations and thus 11 different representatives π_{rp}.

4. Class "4,5,2,6,3"
Analogously as for the third class "4,2,5,3,6", there are 11 different representatives π_{rp}.

5. Class "4,2,5,6,3"
For 1:

1 appears before 4, equivalently after 3, (S.58)

1 appears between 4 and 2. (S.59)

For 7:

7 appears before 4, equivalently after 6, (S.60)

7 appears between 4 and 2, (S.61)

7 appears between 2 and 5,

7 appears between 6 and 3.

Since two orders of 1 and 7 are possible in the combinations (S.58)–(S.60), (S.59)–(S.61), this leads to $2 \cdot 4 + 2 = 10$ combinations and thus 10 different representatives π_{rp}.

6. Class "4,5,2,3,6"

Analogously as for the fifth class "4,2,5,6,3", there are 10 different representatives π_{rp}.

Note, if we allowed 4 to lie between 2 and 3, then according to (S.34) the element 1 would also have to lie between 2 and 3 so that for example we would have $(2, 1, 4, 3, 5, 6, 7)$. However, $(2, 1, 4, 3, 5, 6, 7)$ is a member of the equivalence class $\mathbb{A}_7((1, 4, 3, 5, 6, 7, 2))$ which is included in the class "4,3,5,6,2" which is a subclass of "3,2,5,6".

Hence, in the class "2,3,5,6", there are $19 + 19 + 11 + 11 + 10 + 10 = 80$ different representatives π_{rp} and thus 80 different equivalence classes $A_7(\pi_{rp})$. The same holds for the classes "3,2,5,6", "2,3,6,5", "3,2,6,5" so that altogether there are $4 \cdot 80$ different equivalence classes $\mathbb{A}_7(\pi_{rp})$ where π_{rp} satisfies (S.30). Since each equivalence class has 7 members, we get

$$\sharp(\Pi_7^0) = 4 \cdot 80 \cdot 7 = (4 \cdot 4) \cdot (4 \cdot 5 \cdot 7)$$

as in the first proof of Theorem 2 (e). □

S.3 Application of bivariate simplex depth to testing

Using Theorems 3 and 4, we consider the following standardized versions of the bivariate simplex depths:

$$\begin{aligned}
T_1^S(r_1, \ldots, r_N) &:= \sqrt{N-2}\ \frac{d_1^S(r_1, \ldots, r_N) - \frac{1}{4}}{\frac{1}{4} \cdot \sqrt{\frac{11}{3}}}, \\
T_2^S(r_1, \ldots, r_N) &:= \sqrt{N-3}\ \frac{d_2^S(r_1, \ldots, r_N) - \frac{1}{12}}{\frac{1}{12} \cdot \sqrt{\frac{169}{10}}}, \\
T_1^F(r_1, \ldots, r_N) &:= N\ \left(d_1^F(r_1, \ldots, r_N) - \frac{1}{4}\right), \\
T_2^F(r_1, \ldots, r_N) &:= N\ \left(d_2^F(r_1, \ldots, r_N) - \frac{1}{12}\right).
\end{aligned}$$

Set now

$$\widetilde{q}_{i,j}^N(\alpha) \text{ is } \alpha - \text{quantile of } \{T_i^j(R_1^m, \ldots, R_N^m),\ m = 1, \ldots, M\}$$

for $i = 1, 2$ and $j = S, F$ when $R_1^m, \ldots, R_N^m$ satisfy the assumptions (A_1)–(A_5). Then the tests

$$\text{reject } H_0 \text{ if } \sup_{\theta \in \Theta_0} T_i^j(R_1(\theta), \ldots, R_N(\theta)) < \widetilde{q}_{i,j}^N(\alpha)$$

with $i = 1, 2$ and $j = S, F$ are approximate α-level tests for

$$H_0 : \theta \in \Theta_0 \text{ against } H_0 : \theta \in \Theta_1 = \Theta \setminus \Theta_0.$$

Table S.2 Simulated 5%-quantiles of the bivariate K-simplex depths and their standardized forms for $K = 1, 2$, where always 10^6 simulations runs were used.

Depth	$N = 30$	$N = 100$	Asymptotic value
d_1^S	0.1071429	0.17346939	
d_2^S	< 0	0.03092784	
T_1^S	-1.579084	-1.582605	-1.644854
T_2^S	<-1.263975	-1.506609	-1.644854
d_1^F	0.20689655	0.23755102	
d_2^F	0.06236088	0.07747859	
T_1^F	-1.2931034	-1.2448980	
T_2^F	-0.6291735	-0.5854739	

Table S.2 provides simulated quantiles for $N = 30$ and $N = 100$. For $N = 30$, note that the simplified depths attain at most 28 and 27 values, respectively. In particular, the smallest possible value of the simplified 2-simplex depth has a probability under H_0 which is greater than $\alpha = 0.05$. This is the reason why we write < 0 and < -1.263975 in Table S.2 which means that we can never reject the null hypothesis. To avoid this situation, we also consider randomized tests in the simulation study below. In this case, we assume a one-point hypothesis $H_0 : \theta = \theta_0$. Then the randomized tests are given by

$$\begin{aligned} &\text{reject } H_0 \text{ with probability 1 if } d_K^S(R_1(\theta_0), \ldots, R_N(\theta_0)) < c_K^N(\alpha) \\ &\text{and} \\ &\text{reject } H_0 \text{ with probability } \gamma_K^N(\alpha) \text{ if } d_K^S(R_1(\theta_0), \ldots, R_N(\theta_0)) = c_K^N(\alpha), \end{aligned} \tag{S.62}$$

where $c_K^N(\alpha)$ is the largest value with $\mathbb{P}(d_K^S(R_1(\theta_0), \ldots, R_N(\theta_0)) < c_K^N(\alpha)) \leq \alpha$ and

$$\gamma_K^N(\alpha) := \frac{\alpha - \mathbb{P}(d_K^S(R_1(\theta_0), \ldots, R_N(\theta_0)) < c_K^N(\alpha))}{\mathbb{P}(d_K^S(R_1(\theta_0), \ldots, R_N(\theta_0)) = c_K^N(\alpha))}$$

for $K = 1, 2$. The values $c_K^N(\alpha)$ and $\gamma_K^N(\alpha)$ are given in Table S.3. They were simulated with 10^6 simulation runs. In the simulation study in Section 7 of the main paper, the tests based on the simplified simplex depths are always used in their randomized test version. We compare them with corresponding tests based on the component univariate depths.

S.4 Application of bivariate component depth to testing

We consider the same testing problem as in Section 4 and now use the bivariate component depth notions. For larger sample sizes, it is again useful to use standardized versions of the depths which are converging in distribution.

Table S.3 Simulated values of $c_K^N(\alpha)$ and $\gamma_K^N(\alpha)$ for $\alpha = 5\%$ for the randomized tests based on simplified K-simplex depths and their standardized forms for $K = 1, 2$, where 10^6 simulation runs were used.

Depth	N	$c_K^N(0.05)$	$\gamma_K^N(0.05)$
d_1^S	30	0.107	0.488
T_1^S	100	-1.583	0.385
d_2^S	30	0	0.321
T_2^S	100	-1.507	0.511

Kustosz et al. (2016b) derived the asymptotic distribution of the simplified $(K+1)$-sign depth applied to univariate residuals $R_{1,1}, \ldots, R_{N,1}$ satisfying $\mathbb{P}(R_{n,1} > 0) = \frac{1}{2} = \mathbb{P}(R_{n,1} < 0)$ for $n = 1, \ldots, N$. This is given by

$$\sqrt{N-K}\ \frac{d_{K+1}^{uS}(R_{1,1}, \ldots, R_{N,1}) - \left(\frac{1}{2}\right)^K}{\sqrt{(\frac{1}{2})^K \cdot [3 - (\frac{1}{2})^{K-1} \cdot K - 3 \cdot (\frac{1}{2})^K]}} \longrightarrow \mathcal{N}(0,1).$$

In particular for $K = 1$, the denominator of the standardized depth is the square root of

$$\left(\frac{1}{2}\right)^1 \cdot \left[3 - \left(\frac{1}{2}\right)^{1-1} \cdot 1 - 3 \cdot \left(\frac{1}{2}\right)^1\right] = \frac{1}{2}\left[3 - 1 - \frac{3}{2}\right] = \frac{1}{4},$$

and for $K = 2$, it is

$$\left(\frac{1}{2}\right)^2 \cdot \left[3 - \left(\frac{1}{2}\right)^{2-1} \cdot 2 - 3 \cdot \left(\frac{1}{2}\right)^2\right] = \frac{1}{4}\left[3 - 1 - \frac{3}{4}\right] = \frac{1}{4}\frac{5}{4}.$$

Transferring this result to the simplified component depth of bivariate residuals $r_1 = (r_{1,1}, r_{1,2})^\top, \ldots, r_N = (r_{N,1}, r_{N,2})^\top \in \mathbb{R}^2$ provides the following standardized versions of the simplified component depths (note that the $(K+1)$-sign depth d_{K+1}^{uS} is equivalent to a univariate K-simplex depth d_K^S)

$$\begin{aligned}
T_1^{cS}(r_1, \ldots, r_N) &:= \sqrt{N-1}\ \frac{d_1^{cS}(r_1, \ldots, r_N) - \frac{1}{2}}{\frac{1}{2}} \\
&= \sqrt{N-1}\ \min_{i=1,2} \frac{d_2^{uS}(r_{1,i}, \ldots, r_{N,i}) - \frac{1}{2}}{\frac{1}{2}}, \\
T_2^{cS}(r_1, \ldots, r_N) &:= \sqrt{N-2}\ \frac{d_2^{cS}(r_1, \ldots, r_N) - \frac{1}{4}}{\frac{1}{4} \cdot \sqrt{5}} \\
&= \sqrt{N-2}\ \min_{i=1,2} \frac{d_3^{uS}(r_{1,i}, \ldots, r_{N,i}) - \frac{1}{4}}{\frac{1}{4} \cdot \sqrt{5}}.
\end{aligned}$$

Moreover, Malcherczyk et al. (2021) derived the asymptotic distribution of the standardized versions of the univariate full K-sign depth statistics. Transferring this result to the full component depth as above provides the following standardized versions of the full component depths

$$T_1^{cF}(r_1, \ldots, r_N) := N \left(d_1^{cF}(r_1, \ldots, r_N) - \frac{1}{2} \right),$$
$$T_2^{cF}(r_1, \ldots, r_N) := N \left(d_2^{cF}(r_1, \ldots, r_N) - \frac{1}{4} \right).$$

Hence, the tests for residuals $R_1(\theta) = (R_{1,1}(\theta), R_{1,2}(\theta))^\intercal, \ldots, R_N(\theta) = (R_{N,1}(\theta), R_{N,2}(\theta))^\intercal \in \mathbb{R}^2$ based on the component depths are given as

$$\text{reject } H_0 \quad \text{if} \quad \sup_{\theta \in \Theta_0} d_K^{cj}(R_1(\theta), \ldots, R_N(\theta)) < q_{K,j,c}^N(\alpha)$$

or

$$\text{reject } H_0 \quad \text{if} \quad \sup_{\theta \in \Theta_0} T_K^{cj}(R_1(\theta), \ldots, R_N(\theta)) < \widetilde{q}_{K,j,c}^N(\alpha)$$

for $K = 1, 2$ and $j = S, F$. Here $q_{K,j,c}^N(\alpha)$ and $\widetilde{q}_{K,j,c}^N(\alpha)$ are the α-quantiles of the simulated values of $\{d_K^{cj}(R_1^m, \ldots, R_N^m),\ m = 1, \ldots, M\}$ and $\{T_K^{cj}(R_1^m, \ldots, R_N^m),\ m = 1, \ldots, M\}$, respectively, for $K = 1, 2$ and $j = S, F$, when $R_1^m, \ldots, R_N^m$ satisfy the assumptions (A_1)–(A_5).

We use again the depth statistics d_K^{cj} only for small samples sizes and the standardized depth statistics T_K^{cj} for samples sizes from $N = 100$. The simulated quantiles are given in Table S.4.

We could also apply two tests based on univariate sign depth using Bonferroni adjustment. Then the tests would be

$$\text{reject } H_0 \quad \text{if} \quad \min_{i=1,2} \sup_{\theta \in \Theta_0} d_{K+1}^{uj}(R_{1,i}(\theta), \ldots, R_{N,i}(\theta)) < q_{K,j,u}^N\left(\frac{\alpha}{2}\right)$$

or

$$\text{reject } H_0 \quad \text{if} \quad \min_{i=1,2} \sup_{\theta \in \Theta_0} T_{K+1}^{uj}(R_{1,i}(\theta), \ldots, R_{Ni}(\theta)) < \widetilde{q}_{K,j,u}^N\left(\frac{\alpha}{2}\right)$$

for $K = 1, 2$ and $j = S, F$ where $q_{K,j,u}^N(\alpha)$ and $\widetilde{q}_{K,j,u}^N(\alpha)$ are the α-quantiles of the simulated values of e.g. $\{d_{K+1}^{uj}(R_{1,1}^m, \ldots, R_{N,1}^m),\ m = 1, \ldots, M\}$ and $\{T_{K+1}^{uj}(R_{1,1}^m, \ldots, R_{N,1}^m),\ m = 1, \ldots, M\}$, respectively.

Table S.4 shows that the critical values of tests based on a component depth are only slightly larger than critical values of the two univariate tests with Bonferroni adjustment so that a difference is only visible for the full component 2-depth for $N = 100$. Hence, the power of the component depth tests should behave very similarly.

The simplified component depths have the same problem of too few different values as the simplified simplex depths so that we use again the randomized tests based on simplified component depth given by (S.62) where the values of $c_K^N(0.05)$ and $\gamma_K^N(0.05)$ are given in Table S.5.

Table S.4 Simulated 5%-quantiles of the bivariate component K-depths and their standardized forms and simulated 2.5%-quantiles and 5%-quantiles of the univariate $(K+1)$-sign depth and their standardized forms, always for $K = 1, 2$, where 10^6 simulation runs were used.

	5%-quantile			2.5%-quantile		5%-quantile	
Depth	$N = 30$	$N = 100$	Depth	$N = 30$	$N = 100$	$N = 30$	$N = 100$
d_1^{cS}	0.3103448	0.4040404	d_2^{uS}	0.3103448	0.4040404	0.3448276	0.4141414
d_2^{cS}	0.07142857	0.1428571	d_3^{uS}	0.07142857	0.1428571	0.1071429	0.1632653
T_1^{cS}	-2.042649	-1.909572	T_2^{uS}	-2.042649	-1.909572	-1.671258	-1.708564
T_2^{cS}	-1.690309	-1.897367	T_3^{uS}	-1.690309	-1.897367	-1.352247	-1.535963
d_1^{cF}	0.4344828	0.4806061	d_2^{uF}	0.4344828	0.4806061	0.4597701	0.4848485
d_2^{cF}	0.1931034	0.2334694	d_3^{uF}	0.1931034	0.2333828	0.2068966	0.2376129
T_1^{cF}	-1.965517	-1.939394	T_2^{uF}	-1.965517	-1.939394	-1.206897	-1.515152
T_2^{cF}	-1.706897	-1.653061	T_3^{uF}	-1.706897	-1.661719	-1.293103	-1.238714

Table S.5 Simulated values of $c_K^N(\alpha)$ and $\gamma_K^N(\alpha)$ for $\alpha = 5\%$ for the randomized tests based on simplified component K-depths and their standardized forms for $K = 1, 2$, where 10^6 simulations were used.

Depth	N	$c_K^N(0.05)$	$\gamma_K^N(0.05)$
d_1^{cS}	30	0.3103448	0.7148256
T_1^{cS}	100	-1.909572	0.2678937
d_2^{cS}	30	0.07142857	0.3287176
T_2^{cS}	100	-1.897367	0.7799312

S.5 Explanation of the simulation results for the regression models

First, note that we have $Y_n = R_n$ for $H_0 : \theta = 0$. For calculating expected depth values under H_1, we assume that the variance is so small or θ is so large that the second component $Y_{n,2}$ is always negative on $[-1, 0)$ and always positive on $(0, 1]$ for the linear regression under H_1. Similarly, we assume that $Y_{n,2}$ is always positive on $[-1, -1/3)$ and $(1/3, 1]$ and always negative on $(-1/3, 1/3)$ under H_1 for the first quadratic regression model and that $Y_{n,2}$ is always positive on $[-1, -1/2)$ and $(1/2, 1]$ and always negative on $(-1/2, 1/2)$ under H_1 for the second quadratic regression model. Of course, this strict behaviour of signs of $Y_{n,2}$ is not always satisfied in the simulations. However, this assumption leads to approximate expected depth values under H_1 in the calculations below.

If $Y_{n_1,2}, Y_{n_2,2}, Y_{n_3,2}$ are all positive or all negative, then it is not possible that $\mathbb{1}\{0_2 \in \mathbb{S}(R_{n_1}, R_{n_2}, R_{n_3})\} = \mathbb{1}\{0_2 \in \mathbb{S}(Y_{n_1}, Y_{n_2}, Y_{n_3})\} = 1$ holds.

Hence $Y_{n_1,2}, Y_{n_2,2}, Y_{n_3,2}$ must have different signs. This also means that $\mathbb{1}\{0_2 \in \mathbb{S}(Y_{n_1}, Y_{n_2}, Y_{n_3}) \cap \mathbb{S}(Y_{n_2}, Y_{n_3}, Y_{n_4})\} = 1$ can only hold if $Y_{n_1,2}, Y_{n_2,2}, Y_{n_3,2}$ as well as $Y_{n_2,2}, Y_{n_3,2}, Y_{n_4,2}$ have different signs. The number of these cases for the models is calculated below.

However, first we calculate the conditional probabilities of $\mathbb{1}\{0_2 \in \mathbb{S}(Y_{n_1}, Y_{n_2}, Y_{n_3})\} = 1$ and $\mathbb{1}\{0_2 \in \mathbb{S}(Y_{n_1}, Y_{n_2}, Y_{n_3}) \cap \mathbb{S}(Y_{n_2}, Y_{n_3}, Y_{n_4})\} = 1$ when the signs of $Y_{n_1,2}, Y_{n_2,2}, Y_{n_3,2}, Y_{n_4,2}$ are given. I.e. we calculate the probabilities that appropriate first components $Y_{n_1,1}, Y_{n_2,1}, Y_{n_3,1}, Y_{n_4,1}$ can be found when the signs if $Y_{n_1,2}, Y_{n_2,2}, Y_{n_3,2}, Y_{n_4,2}$ are given.

For simplicity, we consider Y_1, Y_2, Y_3 and let S_1 be the event that $\mathbb{1}\{0_2 \in \mathbb{S}(Y_1, Y_2, Y_3)\} = 1$ is satisfied. Similarly, let S_2 be the event that $\mathbb{1}\{0_2 \in \mathbb{S}(Y_2, Y_3, Y_4)\} = 1$ is satisfied. Define also

$$\begin{aligned}
\Sigma &:= \{-1, 1\}^3 \setminus \{(-1, -1, -1)^\top, (+1, +1, +1)^\top\}, \\
\Sigma_1 &:= \{(-1, +1, -1, -1)^\top, (-1, -1, +1, -1)^\top, \\
&\qquad (+1, -1, +1, +1)^\top, (+1, +1, -1, +1)^\top\}, \\
\Sigma_2 &:= \{(-1, +1, +1, -1)^\top, (+1, -1, -1, +1)^\top\}, \\
\Sigma_3 &:= \{(-1, -1, +1, +1)^\top, (+1, +1, -1, -1)^\top\}, \\
\Sigma_4 &:= \{(-1, +1, -1, +1)^\top, (+1, -1, +1, -1)^\top\},
\end{aligned}$$

$$\operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) := (\operatorname{sign}(Y_{1,2}), \operatorname{sign}(Y_{2,2}), \operatorname{sign}(Y_{3,2}), \operatorname{sign}(Y_{4,2}))^\top,$$

and $\operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2})$ analogously.

Lemma S.5

$$\begin{aligned}
&(a)\ \ \mathbb{P}(S_1 \mid \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}) = s) = \frac{1}{3} \ \textit{for } s \in \Sigma, \\
&(b)\ \ \mathbb{P}(S_1 \cap S_2 \mid \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) = \frac{1}{6} \ \textit{for } s \in \Sigma_1, \\
&(c)\ \ \mathbb{P}(S_1 \cap S_2 \mid \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) = \frac{1}{6} \ \textit{for } s \in \Sigma_2, \\
&(d)\ \ \mathbb{P}(S_1 \cap S_2 \mid \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) = \frac{1}{12} \ \textit{for } s \in \Sigma_3, \\
&(e)\ \ \mathbb{P}(S_1 \cap S_2 \mid \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) = \frac{1}{12} \ \textit{for } s \in \Sigma_4.
\end{aligned}$$

All other conditional probabilities are zero.

Note also that the conditional probabilities of $S_1 \cap S_2$ depend only on whether $Y_{1,2}$ and $Y_{4,2}$ have different signs or not: if $Y_{1,2}$ and $Y_{4,2}$ have the same sign, then the conditional probabilities are $\frac{1}{6}$, and if the signs differ, then the conditional probabilities are $\frac{1}{12}$.

Proof of Lemma S.5. We use again the normalized angles $A_n := \alpha(R_n/\|R_n\|) = \alpha(Y_n/\|Y_n\|)$. Since the signs of the second component are given, we have $A_n \in (0, \frac{1}{2})$ if and only if $\operatorname{sign}(Y_{n,2}) = +1$ and $A_n \in (\frac{1}{2}, 1)$

if and only if $\operatorname{sign}(Y_{n,2}) = -1$. We know from Lemma 3 that $\mathbb{P}(S_1)$ does not depend on the distribution of the normalized angles. Therefore, we can assume that the normalized A_n have a uniform distribution on $[0,1]$.

Part (a). Consider first $s = (-1,+1,+1)^\mathsf{T}$ and note that the uniform distribution of A_n on $[0,1]$ and the independence of A_1, A_2, and A_3 provides

$$\mathbb{P}\left(\left\{A_1 \in \left(\frac{1}{2},1\right)\right\} \cap \left\{A_2 \in \left(0,\frac{1}{2}\right)\right\} \cap \left\{A_3 \in \left(0,\frac{1}{2}\right)\right\}\right) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

Lemma 2 (a) provides

$$\begin{aligned}
&\mathbb{P}(S_1 \mid \operatorname{sign}(Y_{1,2},Y_{2,2},Y_{3,2}) = s) = \mathbb{P}(S_1 | Y_{1,2} < 0, Y_{2,2} > 0, Y_{3,2} > 0)\\
&= \quad \mathbb{P}\left(S_1 | A_1 \in \left(\frac{1}{2},1\right), A_2 \in \left(0,\frac{1}{2}\right), A_3 \in \left(0,\frac{1}{2}\right)\right)\\
&= \quad \frac{\mathbb{P}\left(S_1 \cap \{A_1 \in \left(\frac{1}{2},1\right)\} \cap \{A_2 \in \left(0,\frac{1}{2}\right)\} \cap \{A_3 \in \left(0,\frac{1}{2}\right)\}\right)}{\mathbb{P}\left(\{A_1 \in \left(\frac{1}{2},1\right)\} \cap \{A_2 \in \left(0,\frac{1}{2}\right)\} \cap \{A_3 \in \left(0,\frac{1}{2}\right)\}\right)}\\
&= \quad 8 \cdot \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \mathbb{P}\left(S_1 \cap \left\{A_1 \in \left(\frac{1}{2},1\right)\right\} \Big| A_2 = a_2, A_3 = a_3\right) \, \mathrm{d}\,a_2 \, \mathrm{d}\,a_3\\
&\overset{\text{L. 2 (a)}}{=} \quad 8 \cdot \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \mathbb{P}\left(\min(a_2,a_3) + \frac{1}{2} < A_1 < \max(a_2,a_3) + \frac{1}{2}\right) \, \mathrm{d}\,a_2 \, \mathrm{d}\,a_3\\
&= \quad 8 \cdot \int_0^{\frac{1}{2}} \left[\int_0^{a_3} (a_3 - a_2) \, \mathrm{d}\,a_2 + \int_{a_3}^{\frac{1}{2}} (a_2 - a_3) \, \mathrm{d}\,a_2\right] \mathrm{d}\,a_3\\
&= \quad 8 \cdot \int_0^{\frac{1}{2}} \left[(a_3 a_2 - \frac{1}{2}a_2^2)\Big|_0^{a_3} + (\frac{1}{2}a_2^2 - a_3 a_2)\Big|_{a_3}^{\frac{1}{2}}\right] \mathrm{d}\,a_3\\
&= \quad 8 \cdot \int_0^{\frac{1}{2}} \left[(a_3^2 - \frac{1}{2}a_3^2) + (\frac{1}{8} - a_3\frac{1}{2}) - (\frac{1}{2}a_3^2 - a_3^2)\right] \mathrm{d}\,a_3\\
&= \quad 8 \cdot \int_0^{\frac{1}{2}} \left[a_3^2 - \frac{1}{2}a_3 + \frac{1}{8}\right] \mathrm{d}\,a_3\\
&= \quad 8 \cdot \left[\frac{1}{3}a_3^3 - \frac{1}{2}\frac{1}{2}a_3^2 + \frac{1}{8}a_3\right]\Big|_0^{\frac{1}{2}}\\
&= \quad 8 \cdot \left[\frac{1}{3}\frac{1}{8} - \frac{1}{4}\frac{1}{4} + \frac{1}{8}\frac{1}{2}\right] = \frac{1}{3}.
\end{aligned}$$

Because of symmetry, the assertion also holds for all other $s \in \Sigma$.

Part (b). Consider first $s = (+1,+1,-1,+1)^\mathsf{T} \in \Sigma_1$ and note that the condition $Y_{1,2} > 0, Y_{2,2} > 0, Y_{3,2} < 0, Y_{4,2} > 0$ is equivalent to the condition $A_1 \in \left(0,\frac{1}{2}\right), A_2 \in \left(0,\frac{1}{2}\right), A_3 \in \left(\frac{1}{2},1\right), A_4 \in \left(0,\frac{1}{2}\right)$. We will condition on $A_2 = a_2 \in \left(0,\frac{1}{2}\right)$ and $A_3 = a_3 \in \left(\frac{1}{2},1\right)$ so that we have conditional independence. Then $a_2 < a_3$. If

$$|a_2 - a_3| = a_3 - a_2 > \frac{1}{2}, \tag{S.63}$$

then Lemma 2 (c) provides that the conditional event $S_1 \cap S_2$ given $A_2 = a_2 \in \left(0, \frac{1}{2}\right)$ and $A_3 = a_3 \in \left(\frac{1}{2}, 1\right)$ only holds if

$$a_3 - \frac{1}{2} < A_1 < a_2 + \frac{1}{2},\ a_3 - \frac{1}{2} < A_4 < a_2 + \frac{1}{2}. \tag{S.64}$$

Since we additionally condition on $A_1 \in \left(0, \frac{1}{2}\right), A_4 \in \left(0, \frac{1}{2}\right)$, the upper bounds in (S.64) reduce to $\frac{1}{2}$ so that (S.63) and (S.64) reduce to

$$a_2 < a_3 - \frac{1}{2},\ a_3 - \frac{1}{2} < A_1 < \frac{1}{2},\ a_3 - \frac{1}{2} < A_4 < \frac{1}{2}. \tag{S.65}$$

If

$$|a_2 - a_3| = a_3 - a_2 < \frac{1}{2}, \tag{S.66}$$

then Lemma 2 (d) provides that the conditional event $S_1 \cap S_2$ given $A_2 = a_2 \in \left(0, \frac{1}{2}\right)$ and $A_3 = a_3 \in \left(\frac{1}{2}, 1\right)$ only holds if

$$\begin{aligned} &0 \leq A_1 < a_3 - \frac{1}{2},\ a_2 + \frac{1}{2} < A_1 \leq 1, \\ &0 \leq A_4 < a_3 - \frac{1}{2},\ a_2 + \frac{1}{2} < A_4 \leq 1. \end{aligned} \tag{S.67}$$

Since we additionally condition on $A_1 \in \left(0, \frac{1}{2}\right), A_4 \in \left(0, \frac{1}{2}\right)$, the conditions $a_2 + \frac{1}{2} < A_1 \leq 1$ and $a_2 + \frac{1}{2} < A_4 \leq 1$ are not possible here so that (S.66) and (S.67) reduce to

$$a_2 > a_3 - \frac{1}{2},\ 0 \leq A_1 < a_3 - \frac{1}{2},\ 0 \leq A_4 < a_3 - \frac{1}{2}. \tag{S.68}$$

Similarly as in Case (a), we get with the conditional independence of $S_1 \cap \left\{A_1 \in \left(0, \frac{1}{2}\right)\right\}$ and $S_2 \cap \left\{A_4 \in \left(0, \frac{1}{2}\right)\right\}$ given $A_2 = a_2, A_3 = a_3$ that

$$\begin{aligned}
&\mathbb{P}(S_1 \cap S_2 \mid \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) \\
&= \quad \mathbb{P}(S_1 \cap S_2 | Y_{1,2} > 0, Y_{2,2} > 0, Y_{3,2} < 0, Y_{4,2} > 0) \\
&= \quad \mathbb{P}\left(S_1 \cap S_2 | A_1 \in \left(0, \frac{1}{2}\right), A_2 \in \left(0, \frac{1}{2}\right), A_3 \in \left(\frac{1}{2}, 1\right), A_4 \in \left(0, \frac{1}{2}\right)\right) \\
&= \quad \frac{\mathbb{P}\left(S_1 \cap S_2 \cap \{A_1 \in \left(0, \frac{1}{2}\right)\} \cap \{A_2 \in \left(0, \frac{1}{2}\right)\} \cap \{A_3 \in \left(\frac{1}{2}, 1\right)\} \cap \{A_4 \in \left(0, \frac{1}{2}\right)\}\right)}{\mathbb{P}\left(\{A_1 \in \left(0, \frac{1}{2}\right)\} \cap \{A_2 \in \left(0, \frac{1}{2}\right)\} \cap \{A_3 \in \left(\frac{1}{2}, 1\right)\} \cap \{A_4 \in \left(0, \frac{1}{2}\right)\}\right)} \\
&= \quad 16 \cdot \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} \mathbb{P}\left(S_1 \cap S_2 \cap \left\{A_1 \in \left(0, \frac{1}{2}\right)\right\} \cap \left\{A_4 \in \left(0, \frac{1}{2}\right)\right\} \right. \\
&\qquad \left. \Big| A_2 = a_2, A_3 = a_3\right) \,\mathrm{d}\,a_2 \,\mathrm{d}\,a_3 \\
&= \quad 16 \cdot \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} \mathbb{P}\left(S_1 \cap \left\{A_1 \in \left(0, \frac{1}{2}\right)\right\} \Big| A_2 = a_2, A_3 = a_3\right) \\
&\qquad \mathbb{P}\left(S_2 \cap \left\{A_4 \in \left(0, \frac{1}{2}\right)\right\} \Big| A_2 = a_2, A_3 = a_3\right) \,\mathrm{d}\,a_2 \,\mathrm{d}\,a_3
\end{aligned}$$

$$
\begin{aligned}
&\overset{(S.65),(S.68)}{=} && 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{0}^{a_3-\frac{1}{2}} \mathbb{P}\left(a_3 - \frac{1}{2} < A_1 < \frac{1}{2}\right) \mathbb{P}\left(a_3 - \frac{1}{2} < A_4 < \frac{1}{2}\right) \, \mathrm{d}\, a_2 \right.\\
& && \qquad \left. + \int_{a_3-\frac{1}{2}}^{\frac{1}{2}} \mathbb{P}\left(0 \le A_1 < a_3 - \frac{1}{2}\right) \mathbb{P}\left(0 \le A_4 < a_3 - \frac{1}{2}\right) \, \mathrm{d}\, a_2 \right] \mathrm{d}\, a_3 \\
&= && 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{0}^{a_3-\frac{1}{2}} (1-a_3)^2 \, \mathrm{d}\, a_2 + \int_{a_3-\frac{1}{2}}^{\frac{1}{2}} (a_3 - \frac{1}{2})^2 \, \mathrm{d}\, a_2 \right] \mathrm{d}\, a_3 \\
&= && 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{0}^{a_3-\frac{1}{2}} (1-a_3)^2 \, \mathrm{d}\, a_2 + \int_{a_3-\frac{1}{2}}^{\frac{1}{2}} (a_3 - 1 + \frac{1}{2})^2 \, \mathrm{d}\, a_2 \right] \mathrm{d}\, a_3 \\
&= && 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{0}^{a_3-\frac{1}{2}} (1-a_3)^2 \, \mathrm{d}\, a_2 + \int_{a_3-\frac{1}{2}}^{\frac{1}{2}} (a_3 - 1)^2 + (a_3-1) + \frac{1}{4} \, \mathrm{d}\, a_2 \right] \mathrm{d}\, a_3 \\
&= && 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{0}^{\frac{1}{2}} (1-a_3)^2 \, \mathrm{d}\, a_2 + \int_{a_3-\frac{1}{2}}^{\frac{1}{2}} (a_3 - 1) + \frac{1}{4} \, \mathrm{d}\, a_2 \right] \mathrm{d}\, a_3 \\
&= && 16 \cdot \int_{\frac{1}{2}}^{1} \left[\frac{1}{2}(1-a_3)^2 + (1-a_3)\left((a_3-1) + \frac{1}{4}\right) \right] \mathrm{d}\, a_3 \\
&= && 16 \cdot \int_{\frac{1}{2}}^{1} \left[\frac{1}{2}(1-a_3)^2 - (a_3-1)^2 - \frac{1}{4}(a_3-1) \right] \mathrm{d}\, a_3 \\
&= && 16 \cdot \int_{\frac{1}{2}}^{1} \left[-\frac{1}{2}(1-a_3)^2 - \frac{1}{4}(a_3-1) \right] \mathrm{d}\, a_3 \\
&= && 16 \cdot \int_{\frac{1}{2}}^{1} \left[-\frac{1}{2}(a_3^2 - 2a_3 + 1) - \frac{1}{4}(a_3-1) \right] \mathrm{d}\, a_3 \\
&= && 16 \cdot \int_{\frac{1}{2}}^{1} \left[-\frac{1}{2}a_3^2 + a_3 - \frac{1}{2} - \frac{1}{4}a_3 + \frac{1}{4} \right] \mathrm{d}\, a_3 \\
&= && 16 \cdot \int_{\frac{1}{2}}^{1} \left[-\frac{1}{2}a_3^2 + \frac{3}{4}a_3 - \frac{1}{4} \right] \mathrm{d}\, a_3 = 8 \cdot \int_{\frac{1}{2}}^{1} \left[-a_3^2 + \frac{3}{2}a_3 - \frac{1}{2} \right] \mathrm{d}\, a_3 \\
&= && 8 \cdot \left[-\frac{1}{3}a_3^3 + \frac{3}{2}\frac{1}{2}a_3^2 - \frac{1}{2}a_3 \Big|_{\frac{1}{2}}^{1} \right] \\
&= && 8 \cdot \left[-\frac{1}{3} + \frac{3}{2}\frac{1}{2} - \frac{1}{2} - \left(-\frac{1}{3}\frac{1}{8} + \frac{3}{4}\frac{1}{4} - \frac{1}{4} \right) \right] \\
&= && 8 \cdot \left[-\frac{1}{3} + \frac{3}{4} - \frac{1}{2} + \frac{1}{3}\frac{1}{8} - \frac{3}{4}\frac{1}{4} + \frac{1}{4} \right] \\
&= && 8 \cdot \frac{-16 + 36 - 24 + 2 - 9 + 12}{3 \cdot 4 \cdot 4} \\
&= && 8 \cdot \frac{-40 + 48 - 7}{3 \cdot 4 \cdot 4} = 8 \cdot \frac{1}{3 \cdot 4 \cdot 4} = \frac{1}{6}.
\end{aligned}
$$

The assertion for other $s \in \Sigma_1$ follows by symmetry and by interchanging the role of $Y_{2,2}$ and $Y_{3,2}$ or A_2 and A_3, respectively.

Part (c). Consider first $s = (+1,-1,-1,+1)^{\mathsf{T}} \in \Sigma_2$. Here the condition $Y_{1,2} > 0, Y_{2,2} < 0, Y_{3,2} < 0, Y_{4,2} > 0$ is equivalent to the condition $A_1 \in \left(0, \frac{1}{2}\right), A_2 \in \left(\frac{1}{2}, 1\right), A_3 \in \left(\frac{1}{2}, 1\right), A_4 \in \left(0, \frac{1}{2}\right)$. We will condition on $A_2 = a_2 \in \left(\frac{1}{2}, 1\right)$ and $A_3 = a_3 \in \left(\frac{1}{2}, 1\right)$ so that we have conditional independence. Here we are in the situation of Lemma 2 (b) so that we get similar as in Part (b)

$$
\begin{aligned}
&\mathbb{P}(S_1 \cap S_2 \mid \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) \\
&\quad = \quad \mathbb{P}(S_1 \cap S_2 | Y_{1,2} > 0, Y_{2,2} < 0, Y_{3,2} < 0, Y_{4,2} > 0) \\
&\quad = \quad \mathbb{P}\left(S_1 \cap S_2 | A_1 \in \left(0, \frac{1}{2}\right), A_2 \in \left(\frac{1}{2}, 1\right), A_3 \in \left(\frac{1}{2}, 1\right), A_4 \in \left(0, \frac{1}{2}\right)\right) \\
&\quad = \quad 16 \cdot \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \mathbb{P}\left(S_1 \cap S_2 \cap \left\{A_1 \in \left(0, \frac{1}{2}\right)\right\} \cap \left\{A_4 \in \left(0, \frac{1}{2}\right)\right\}\right. \\
&\qquad\qquad\qquad\qquad \left. \Big| A_2 = a_2, A_3 = a_3\right) \,\mathrm{d}\, a_2 \,\mathrm{d}\, a_3 \\
&\quad = \quad 16 \cdot \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \mathbb{P}\left(S_1 \cap \left\{A_1 \in \left(0, \frac{1}{2}\right)\right\} \Big| A_2 = a_2, A_3 = a_3\right) \\
&\qquad\qquad\qquad \mathbb{P}\left(S_2 \cap \left\{A_4 \in \left(0, \frac{1}{2}\right)\right\} \Big| A_2 = a_2, A_3 = a_3\right) \,\mathrm{d}\, a_2 \,\mathrm{d}\, a_3 \\
&\overset{\text{L. 2 (b)}}{=} \quad 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{\frac{1}{2}}^{a_3} \mathbb{P}\left(a_2 - \frac{1}{2} < A_1 < a_3 - \frac{1}{2}\right)\right. \\
&\qquad\qquad\qquad \cdot \mathbb{P}\left(a_2 - \frac{1}{2} < A_4 < a_3 - \frac{1}{2}\right) \,\mathrm{d}\, a_2 \\
&\qquad\qquad + \int_{a_3}^{1} \mathbb{P}\left(a_3 - \frac{1}{2} \le A_1 < a_2 - \frac{1}{2}\right) \\
&\qquad\qquad\qquad \left. \cdot \mathbb{P}\left(a_3 - \frac{1}{2} \le A_4 < a_2 - \frac{1}{2}\right) \,\mathrm{d}\, a_2\right] \,\mathrm{d}\, a_3 \\
&\quad = \quad 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{\frac{1}{2}}^{a_3} (a_3 - a_2)^2 \,\mathrm{d}\, a_2 + \int_{a_3}^{1} (a_2 - a_3)^2 \,\mathrm{d}\, a_2\right] \,\mathrm{d}\, a_3 \\
&\quad = \quad 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{\frac{1}{2}}^{1} (a_3 - a_2)^2 \,\mathrm{d}\, a_2\right] \,\mathrm{d}\, a_3 \\
&\quad = \quad 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{\frac{1}{2}}^{1} \left(a_3^2 - 2a_3a_2 + a_2^2\right) \,\mathrm{d}\, a_2\right] \,\mathrm{d}\, a_3 \\
&\quad = \quad 16 \cdot \int_{\frac{1}{2}}^{1} \left[\left(a_3^2 a_2 - 2a_3 \frac{1}{2} a_2^2 + \frac{1}{3} a_2^3\right) \Big|_{\frac{1}{2}}^{1}\right] \,\mathrm{d}\, a_3 \\
&\quad = \quad 16 \cdot \int_{\frac{1}{2}}^{1} \left[\left(a_3^2 - a_3 + \frac{1}{3}\right) - \left(a_3^2 \frac{1}{2} - a_3 \frac{1}{4} + \frac{1}{3}\frac{1}{8}\right)\right] \,\mathrm{d}\, a_3
\end{aligned}
$$

$$
\begin{aligned}
&= 16 \cdot \int_{\frac{1}{2}}^{1} \left[\frac{1}{2}a_3^2 - \frac{3}{4}a_3 + \frac{1}{3} - \frac{1}{3}\frac{1}{8}\right] \mathrm{d}\, a_3 \\
&= 16 \cdot \left[\frac{1}{2}\frac{1}{3}a_3^3 - \frac{3}{4}\frac{1}{2}a_3^2 + \left(\frac{1}{3} - \frac{1}{3}\frac{1}{8}\right) a_3\right] \Big|_{\frac{1}{2}}^{1} \\
&= 16 \cdot \left[\frac{1}{2}\frac{1}{3} - \frac{3}{8} + \frac{1}{3} - \frac{1}{3}\frac{1}{8} - \left(\frac{1}{2}\frac{1}{3}\frac{1}{8} - \frac{3}{8}\frac{1}{4} + \left(\frac{1}{3} - \frac{1}{3}\frac{1}{8}\right)\frac{1}{2}\right)\right] \\
&= 16 \cdot \left[\frac{1}{2}\frac{1}{3} - \frac{3}{8} + \frac{1}{3} - \frac{1}{3}\frac{1}{8} - \frac{1}{2}\frac{1}{3}\frac{1}{8} + \frac{3}{8}\frac{1}{4} - \frac{1}{3}\frac{1}{2} + \frac{1}{3}\frac{1}{8}\frac{1}{2}\right] \\
&= 16 \cdot \left[-\frac{3}{8} + \frac{1}{3} - \frac{1}{3}\frac{1}{8} + \frac{3}{8}\frac{1}{4}\right] \\
&= 16 \cdot \frac{-36 + 32 - 4 + 9}{4 \cdot 8 \cdot 3} = 16 \cdot \frac{-40 + 41}{4 \cdot 8 \cdot 3} = \frac{1}{6}.
\end{aligned}
$$

The assertion for the other $s \in \Sigma_2$ follows by symmetry.

Part (d). Consider first $s = (+1, +1, -1, -1)^\mathsf{T} \in \Sigma_3$. Here the condition $Y_{1,2} > 0, Y_{2,2} > 0, Y_{3,2} < 0, Y_{4,2} < 0$ is equivalent to the condition $A_1 \in \left(0, \frac{1}{2}\right), A_2 \in \left(0, \frac{1}{2}\right), A_3 \in \left(\frac{1}{2}, 1\right), A_4 \in \left(\frac{1}{2}, 1\right)$. We will condition on $A_2 = a_2 \in \left(0, \frac{1}{2}\right)$ and $A_3 = a_3 \in \left(\frac{1}{2}, 1\right)$ so that we have conditional independence. Then $a_2 < a_3$. If

$$|a_2 - a_3| = a_3 - a_2 > \frac{1}{2}, \tag{S.69}$$

then Lemma 2 (c) provides that the conditional event $S_1 \cap S_2$ given $A_2 = a_2 \in \left(0, \frac{1}{2}\right)$ and $A_3 = a_3 \in \left(\frac{1}{2}, 1\right)$ only holds if

$$a_3 - \frac{1}{2} < A_1 < a_2 + \frac{1}{2},\ a_3 - \frac{1}{2} < A_4 < a_2 + \frac{1}{2}. \tag{S.70}$$

Since we condition additionally on $A_1 \in \left(0, \frac{1}{2}\right), A_4 \in \left(\frac{1}{2}, 1\right)$, the upper bound in (S.70) for A_1 as well as the lower bound in (S.70) for A_1 reduce to $\frac{1}{2}$ so that (S.69) and (S.70) reduce to

$$a_2 < a_3 - \frac{1}{2},\ a_3 - \frac{1}{2} < A_1 < \frac{1}{2},\ \frac{1}{2} < A_4 < a_2 + \frac{1}{2}. \tag{S.71}$$

If

$$|a_2 - a_3| = a_3 - a_2 < \frac{1}{2}, \tag{S.72}$$

then Lemma 2 (d) provides that the conditional event $S_1 \cap S_2$ given $A_2 = a_2 \in \left(0, \frac{1}{2}\right)$ and $A_3 = a_3 \in \left(\frac{1}{2}, 1\right)$ only holds if

$$
\begin{aligned}
&0 \le A_1 < a_3 - \frac{1}{2},\ a_2 + \frac{1}{2} < A_1 \le 1, \\
&0 \le A_4 < a_3 - \frac{1}{2},\ a_2 + \frac{1}{2} < A_4 \le 1.
\end{aligned} \tag{S.73}
$$

Since we condition additionally on $A_1 \in \left(0, \frac{1}{2}\right), A_4 \in \left(\frac{1}{2}, 1\right)$, the conditions $a_2 + \frac{1}{2} < A_1 \leq 1$ and $0 \leq A_4 < a_3 - \frac{1}{2}$ are not possible here so that (S.72) and (S.73) reduce to

$$a_2 > a_3 - \frac{1}{2},\ 0 \leq A_1 < a_3 - \frac{1}{2},\ a_2 + \frac{1}{2} < A_4 \leq 1. \tag{S.74}$$

Similarly as in Case (b), we get with the conditional independence of $S_1 \cap \left\{A_1 \in \left(0, \frac{1}{2}\right)\right\}$ and $S_2 \cap \left\{A_4 \in \left(0, \frac{1}{2}\right)\right\}$ given $A_2 = a_2, A_3 = a_3$

$$\begin{aligned}
&\mathbb{P}(S_1 \cap S_2 \mid \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s)\\
&= \mathbb{P}(S_1 \cap S_2 | Y_{1,2} > 0, Y_{2,2} > 0, Y_{3,2} < 0, Y_{4,2} < 0)\\
&= \mathbb{P}\left(S_1 \cap S_2 | A_1 \in \left(0, \frac{1}{2}\right), A_2 \in \left(0, \frac{1}{2}\right), A_3 \in \left(\frac{1}{2}, 1\right), A_4 \in \left(\frac{1}{2}, 1\right)\right)\\
&= 16 \cdot \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} \mathbb{P}\left(S_1 \cap S_2 \cap \left\{A_1 \in \left(0, \frac{1}{2}\right)\right\} \cap \left\{A_4 \in \left(\frac{1}{2}, 1\right)\right\}\right.\\
&\qquad\qquad \left.\Big| A_2 = a_2, A_3 = a_3\right) \mathrm{d}\, a_2 \, \mathrm{d}\, a_3\\
&= 16 \cdot \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} \mathbb{P}\left(S_1 \cap \left\{A_1 \in \left(0, \frac{1}{2}\right)\right\} \Big| A_2 = a_2, A_3 = a_3\right)\\
&\qquad\qquad \mathbb{P}\left(S_2 \cap \left\{A_4 \in \left(\frac{1}{2}, 1\right)\right\} \Big| A_2 = a_2, A_3 = a_3\right) \mathrm{d}\, a_2 \, \mathrm{d}\, a_3\\
&\overset{(S.71),(S.74)}{=} 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{0}^{a_3 - \frac{1}{2}} \mathbb{P}\left(a_3 - \frac{1}{2} < A_1 < \frac{1}{2}\right) \mathbb{P}\left(\frac{1}{2} < A_4 < a_2 + \frac{1}{2}\right) \mathrm{d}\, a_2\right.\\
&\qquad \left. + \int_{a_3 - \frac{1}{2}}^{\frac{1}{2}} \mathbb{P}\left(0 \leq A_1 < a_3 - \frac{1}{2}\right) \mathbb{P}\left(a_2 + \frac{1}{2} < A_4 \leq 1\right) \mathrm{d}\, a_2\right] \mathrm{d}\, a_3\\
&= 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{0}^{a_3 - \frac{1}{2}} (1 - a_3) a_2 \, \mathrm{d}\, a_2\right.\\
&\qquad \left. + \int_{a_3 - \frac{1}{2}}^{\frac{1}{2}} \left(a_3 - \frac{1}{2}\right)\left(\frac{1}{2} - a_2\right) \mathrm{d}\, a_2\right] \mathrm{d}\, a_3\\
&= 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{0}^{a_3 - \frac{1}{2}} (1 - a_3) a_2 \, \mathrm{d}\, a_2\right.\\
&\qquad \left. + \int_{a_3 - \frac{1}{2}}^{\frac{1}{2}} \left(1 - a_3 - \frac{1}{2}\right)\left(a_2 - \frac{1}{2}\right) \mathrm{d}\, a_2\right] \mathrm{d}\, a_3\\
&= 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{0}^{a_3 - \frac{1}{2}} (1 - a_3) a_2 \, \mathrm{d}\, a_2\right.\\
&\qquad \left. + \int_{a_3 - \frac{1}{2}}^{\frac{1}{2}} \left((1 - a_3) a_2 - \frac{1}{2} a_2 - (1 - a_3)\frac{1}{2} + \frac{1}{4}\right) \mathrm{d}\, a_2\right] \mathrm{d}\, a_3
\end{aligned}$$

$$
\begin{aligned}
= \quad & 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{0}^{\frac{1}{2}} (1-a_3)a_2 \,\mathrm{d}\, a_2 \right. \\
& \left. + \int_{a_3-\frac{1}{2}}^{\frac{1}{2}} \left(-\frac{1}{2}a_2 - \frac{1}{2} + a_3\frac{1}{2} + \frac{1}{4} \right) \,\mathrm{d}\, a_2 \right] \mathrm{d}\, a_3 \\
= \quad & 16 \cdot \int_{\frac{1}{2}}^{1} \left[\int_{0}^{\frac{1}{2}} (1-a_3)a_2 \,\mathrm{d}\, a_2 \right. \\
& \left. + \int_{a_3-\frac{1}{2}}^{\frac{1}{2}} \left(-\frac{1}{2}a_2 - \frac{1}{4} + a_3\frac{1}{2} \right) \,\mathrm{d}\, a_2 \right] \mathrm{d}\, a_3 \\
= \quad & 16 \cdot \int_{\frac{1}{2}}^{1} \left[(1-a_3)\frac{1}{2}a_2^2 \Big|_0^{\frac{1}{2}} \right. \\
& \left. + \left(-\frac{1}{4}a_2^2 - \frac{1}{4}a_2 + a_3\frac{1}{2}a_2 \right) \Big|_{a_3-\frac{1}{2}}^{\frac{1}{2}} \right] \mathrm{d}\, a_3 \\
= \quad & 16 \cdot \int_{\frac{1}{2}}^{1} \left[(1-a_3)\frac{1}{8} + \left(-\frac{1}{16} - \frac{1}{8} + a_3\frac{1}{4} \right) \right. \\
& \left. - \left(-\frac{1}{4}\left(a_3 - \frac{1}{2}\right)^2 - \frac{1}{4}\left(a_3 - \frac{1}{2}\right) + a_3\frac{1}{2}\left(a_3 - \frac{1}{2}\right) \right) \right] \mathrm{d}\, a_3 \\
= \quad & 16 \cdot \int_{\frac{1}{2}}^{1} \left[\frac{1}{8} - a_3\frac{1}{8} - \frac{1}{16} - \frac{1}{8} + a_3\frac{1}{4} \right. \\
& \left. + \frac{1}{4}\left(a_3^2 - a_3 + \frac{1}{4} \right) + \frac{1}{4}a_3 - \frac{1}{8} - \frac{1}{2}a_3^2 + \frac{1}{4}a_3 \right] \mathrm{d}\, a_3 \\
= \quad & 16 \cdot \int_{\frac{1}{2}}^{1} \left[a_3\frac{1}{8} - \frac{1}{16} - \frac{1}{4}a_3^2 - \frac{1}{16} + \frac{1}{4}a_3 \right] \mathrm{d}\, a_3 \\
= \quad & 16 \cdot \int_{\frac{1}{2}}^{1} \left[-\frac{1}{4}a_3^2 + a_3\frac{3}{8} - \frac{1}{8} \right] \mathrm{d}\, a_3 \\
= \quad & 16 \cdot \left[-\frac{1}{4}\frac{1}{3}a_3^3 + \frac{1}{2}a_3^2\frac{3}{8} - \frac{1}{8}a_3 \right] \Big|_{\frac{1}{2}}^{1} \\
= \quad & 16 \cdot \left[-\frac{1}{4}\frac{1}{3} + \frac{1}{2}\frac{3}{8} - \frac{1}{8} - \left(-\frac{1}{4}\frac{1}{3}\frac{1}{8} + \frac{1}{2}\frac{1}{4}\frac{3}{8} - \frac{1}{8}\frac{1}{2} \right) \right] \\
= \quad & 16 \cdot \left[-\frac{1}{4 \cdot 3} + \frac{3}{2 \cdot 8} - \frac{1}{8} + \frac{1}{4 \cdot 3 \cdot 8} - \frac{3}{2 \cdot 4 \cdot 8} + \frac{1}{8 \cdot 2} \right] \\
= \quad & 4 \cdot \left[-\frac{1}{3} + \frac{3}{4} - \frac{1}{2} + \frac{1}{3 \cdot 8} - \frac{3}{2 \cdot 8} + \frac{1}{4} \right] \\
= \quad & 4 \cdot \frac{-16 + 36 - 24 + 2 - 9 + 12}{4 \cdot 4 \cdot 3} = \frac{-40 + 48 - 7}{4 \cdot 3} = \frac{1}{12}.
\end{aligned}
$$

The assertion for the other $s \in \Sigma_3$ follows by symmetry.

Part (e). This assertion follows from Part (d) by interchanging the role of $Y_{2,2}$ and $Y_{3,2}$ or A_2 and A_3, respectively. □

Note that the proof of Lemma S.5 bases on the assumption that the normalized angles A_n have a uniform distribution on $[0,1]$. This might not be satisfied under the alternatives considered below. However, it might be considered as a first approximation of what might happen under the alternatives when signs of the second components are given.

Alternative proofs of Theorem 2 (a) and (b) using Lemma S.5.
(also for checking that Lemma S.5 is correct)

Part (a). There are 8 constellations of signs of $Y_{1,2}, Y_{2,2}, Y_{3,2}$. Two of them, namely $+++$ and $---$, lead to conditional probabilities which are zero. With Lemma S.5 (a), we get

$$\begin{aligned}
&\mathbb{P}(S_1) \\
&= \sum_{s\in\Sigma} \mathbb{P}(S_1 \mid \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}) = s) \cdot \mathbb{P}(\operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}) = s) \\
&= 6 \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{4}.
\end{aligned}$$

Part (b). There are 16 constellations of signs of $Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}$. Six of them, namely $++++$, $----$, $+++-$, $+---$, $---+$, $-+++$, lead to conditional probabilities of $S_1 \cap S_2$ which are zero. Lemma S.5 (b) and (c) provides

$$\mathbb{P}(S_1 \cap S_2 | \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) = \frac{1}{6}$$

for all $s \in \Sigma_1 \cup \Sigma_2$. And Lemma S.5 (d) and (e) provides

$$\mathbb{P}(S_1 \cap S_2 | \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) = \frac{1}{12}$$

for all $s \in \Sigma_3 \cup \Sigma_4$. Hence we get

$$\begin{aligned}
&\mathbb{P}(S_1 \cap S_2) \\
&= \sum_{s\in\Sigma_1\cup\Sigma_2} \mathbb{P}(S_1 \cap S_2 | \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) \\
&\qquad\qquad\qquad \cdot \mathbb{P}(\operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) \\
&+ \sum_{s\in\Sigma_3\cup\Sigma_4} \mathbb{P}(S_1 \cap S_2 | \operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) \\
&\qquad\qquad\qquad \cdot \mathbb{P}(\operatorname{sign}(Y_{1,2}, Y_{2,2}, Y_{3,2}, Y_{4,2}) = s) \\
&= 6 \cdot \frac{1}{6} \cdot \left(\frac{1}{2}\right)^4 + 4 \cdot \frac{1}{12} \cdot \left(\frac{1}{2}\right)^4 = 3 \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^4 + 1 \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^4 \\
&= \frac{4}{3} \cdot \frac{1}{16} = \frac{1}{12}.
\end{aligned}$$

□

Linear regression before rotation. Here we assume that the first $N/2$ signs of the second component $Y_{n,2}$ are positive, and the last $N/2$ signs are negative under the alternative.

For the bivariate full 1-simplex depth, only triples are counted with the sign constellations $++-$ and $+--$ of the second component. The relative number of these triples for each of these two cases is given by

$$\frac{1}{\binom{N}{3}}\binom{N/2}{2}\cdot\frac{N}{2}=\frac{3\cdot 2\cdot\frac{N}{2}\left(\frac{N}{2}-1\right)N}{N\,(N-1)\,(N-2)\,2\cdot 2}=\frac{N\,(N-2)\cdot 3}{8\,(N-1)\,(N-2)}=\frac{3\,N}{8\,(N-1)}.$$

Hence the relative amount of triples where 0_2 can be included in the simplex is

$$2\cdot\frac{3\,N}{8\,(N-1)}=\frac{3\,N}{4\,(N-1)}.$$

This quantity must be multiplied with the probability that appropriate first components $Y_{n,1}$ can be found, which is given by Lemma S.5 (a) as $\frac{1}{3}$. So, we expect approximately

$$\frac{1}{3}\frac{3\,N}{4\,(N-1)}=\frac{N}{4\,(N-1)}\overset{N\to\infty}{\longrightarrow}\frac{1}{4}$$

simplices containing 0_2 under the alternative which converges with increasing N to the expected number under H_0. We have

$$\frac{N}{4\,(N-1)}=\begin{cases}0.2586207 & \text{for } N=30,\\ 0.2525253 & \text{for } N=100.\end{cases}$$

This explains why we cannot reject H_0 with the full 1-simplex depth.

For the bivariate full 2-simplex depth, only quadruples can be counted with the sign constellations $++--$. The relative number of these quadruples is given by

$$\begin{aligned}\frac{1}{\binom{N}{4}}\binom{N/2}{2}\cdot\binom{N/2}{2}&=\frac{1}{\binom{N}{4}}\left(\frac{\frac{N}{2}\left(\frac{N}{2}-1\right)}{2}\right)^2\\ &=\frac{4\cdot 3\cdot 2\cdot N^2\,(N-2)^2}{N\,(N-1)\,(N-2)\,(N-3)\cdot 4\cdot 4\cdot 4}=\frac{N\,(N-2)\cdot 3}{(N-1)\,(N-3)\cdot 8}.\end{aligned}$$

This quantity must be multiplied with the probability that appropriate first components $Y_{n,1}$ can be found which is given by Lemma S.5 (d) as $\frac{1}{12}$ so that we expect approximately

$$\frac{1}{12}\frac{N\,(N-2)\cdot 3}{(N-1)\,(N-3)\cdot 8}=\frac{N\,(N-2)}{(N-1)\,(N-3)\cdot 32}\overset{N\to\infty}{\longrightarrow}\frac{1}{32}=0.03125<\frac{1}{12}$$

pairs of simplices containing 0_2 under the alternative. We have

$$\frac{N\,(N-2)}{(N-1)\,(N-3)\cdot 32} = \begin{cases} 0.0335249 & \text{for } N = 30, \\ 0.0318911 & \text{for } N = 100. \end{cases}$$

That is much less than the expected number of $\frac{1}{12}$ under H_0. Note also that the 5%-quantiles are 0.06236088 and 0.07747859 for $N = 30$ and $N = 100$, respectively, according to Table S.2. Hence the chance for rejection of H_0 is high for $N = 30$ and $N = 100$.

This explains the results in Figure 4.

Quadratic regression: first model. Here we assume that the first $N/3$ signs of the second component $Y_{n,2}$ are positive, the second $N/3$ signs are negative, and the last $N/3$ are positive under the alternative.

For the bivariate full 1-simplex depth, only triples are counted with the sign constellations $++-$, $+--$, $--+$, $-++$, and $+-+$ of the second component. The relative number of these triples for each of the first four constellations is given by

$$\begin{aligned} \frac{1}{\binom{N}{3}} \binom{N/3}{2} \cdot \frac{N}{3} &= \frac{1}{\binom{N}{3}} \frac{\frac{N}{3}\left(\frac{N}{3}-1\right)}{2} \cdot \frac{N}{3} \\ &= \frac{3\cdot 2\cdot N\,(N-3)\,N}{N\,(N-1)\,(N-2)\cdot 3\cdot 3\cdot 2\cdot 3} = \frac{(N-3)\,N}{(N-1)\,(N-2)\cdot 9}. \end{aligned}$$

The relative number of the triples for the last constellation is given by

$$\frac{1}{\binom{N}{3}} \left(\frac{N}{3}\right)^3 = \frac{3\cdot 2\cdot N^3}{N\,(N-1)\,(N-2)\cdot 3\cdot 3\cdot 3} = \frac{N^2\cdot 2}{(N-1)\,(N-2)\cdot 9}.$$

Hence the relative amount of triples where 0_2 can be included in the simplex is

$$\begin{aligned} &4\,\frac{(N-3)\,N}{(N-1)\,(N-2)\cdot 9} + \frac{N^2\cdot 2}{(N-1)\,(N-2)\cdot 9} \\ &= \frac{2\,N}{(N-1)\,(N-2)\cdot 9}\,[2\,(N-3)+N] = \frac{2\,N}{(N-1)\,(N-2)\cdot 9}\,[3\,N-6] \\ &= \frac{2\,N\cdot 3}{(N-1)\,(N-2)\cdot 9}[N-2] = \frac{N\cdot 2}{(N-1)\cdot 3}. \end{aligned}$$

This quantity must be multiplied with the probability that appropriate first components $Y_{n,1}$ can be found, which is given by Lemma S.5 (a) as $\frac{1}{3}$ so that we expect approximately

$$\frac{1}{3}\,\frac{N\cdot 2}{(N-1)\cdot 3} = \frac{N\cdot 2}{(N-1)\cdot 9} \overset{N\to\infty}{\longrightarrow} \frac{2}{9} = 0.2222222 < \frac{1}{4}$$

simplices containing 0_2 under the alternative. We have

$$\frac{N\cdot 2}{(N-1)\cdot 9}=\begin{cases}0.2298851 & \text{for } N=30,\\ 0.2244669 & \text{for } N=100,\end{cases}$$

which is less than the expected number under H_0. Note that 0.20689655 and 0.23755102 are the 5%-quantiles for $N=30$ and $N=100$ according to Table S.2. Hence the rejection rate is high for $N=100$. That already the rejection rate for $N=30$ is quite good, might be explained by the small difference of the approximated expected value and the 5%-quantile.

For the bivariate full 2-simplex depth, only quadruples can be counted with the sign constellations $++--$, $--++$, $++-+$, $+-++$, $+--+$. The relative number of these quadruples of the first two constellations is given by

$$\begin{aligned}&\frac{1}{\binom{N}{4}}\binom{N/3}{2}^2=\frac{1}{\binom{N}{4}}\left(\frac{\frac{N}{3}\left(\frac{N}{3}-1\right)}{2}\right)^2\\ &=\frac{4\cdot 3\cdot 2\cdot N^2\,(N-3)^2}{N\,(N-1)\,(N-2)\,(N-3)\cdot 3^2\cdot 3^2\cdot 2^2}=\frac{2\cdot N\,(N-3)}{(N-1)\,(N-2)\cdot 3^2\cdot 3}.\end{aligned}$$

The relative number of the triples for each of the last three constellations is given by

$$\begin{aligned}&\frac{1}{\binom{N}{4}}\binom{N/3}{2}\cdot\left(\frac{N}{3}\right)^2=\frac{1}{\binom{N}{4}}\frac{\frac{N}{3}\left(\frac{N}{3}-1\right)}{2}\cdot\frac{N^2}{3\cdot 3}\\ &=\frac{4\cdot 3\cdot 2\cdot N\,(N-3)\,N^2}{N\,(N-1)\,(N-2)\,(N-3)\cdot 3^2\cdot 2\cdot 3^2}=\frac{N^2\cdot 4}{(N-1)\,(N-2)\cdot 3^2\cdot 3}.\end{aligned}$$

Both relative numbers must be multiplied with the probability that appropriate first components $Y_{n,1}$ can be found. This probability is $\frac{1}{12}$ for the first two constellations according to Lemma S.5 (d) and $\frac{1}{6}$ for the last three constellations according to Lemma S.5 (b) and (c). Hence we expect approximately

$$\begin{aligned}&2\cdot\frac{1}{12}\,\frac{2\cdot N\,(N-3)}{(N-1)\,(N-2)\cdot 3^2\cdot 3}+3\cdot\frac{1}{6}\,\frac{N^2\cdot 4}{(N-1)\,(N-2)\cdot 3^2\cdot 3}\\ &=\frac{N\cdot 4}{(N-1)\,(N-2)\cdot 3^2\cdot 3\cdot 6}\left[(N-3)\frac{1}{2}+3\,N\right]\\ &=\frac{N}{(N-1)\,(N-2)\cdot 3^2\cdot 3^2}\left[(N-3)+6\,N\right]\\ &=\frac{N\,(7N-3)}{(N-1)\,(N-2)\cdot 9\cdot 9}\xrightarrow{N\to\infty}\frac{7}{81}=0.08641975>\frac{1}{12}=0.08333333\end{aligned}$$

pairs of simplices containing 0_2 under the alternative. We have

$$\frac{N\,(7N-3)}{(N-1)\,(N-2)\cdot 9\cdot 9}=\begin{cases}0.09441708 & \text{for } N=30,\\ 0.08869242 & \text{for } N=100,\end{cases}$$

which is larger than the expected number of $\frac{1}{12}$ under H_0. This explains why the full 2-simplex depth cannot reject H_0 under the alternative.

Note, that for the univariate full 1-simplex depth (the full 2-sign depth), the relative number of pairs with alternating signs for the second component is

$$\frac{\left(\frac{N}{3}\right)^2+\left(\frac{N}{3}\right)^2}{\binom{N}{2}} \stackrel{N\to\infty}{\longrightarrow} \frac{4}{3^2}=\frac{4}{9}=0.4444444<\frac{1}{2}.$$

Further,

$$\frac{\left(\frac{N}{3}\right)^2+\left(\frac{N}{3}\right)^2}{\binom{N}{2}}=\begin{cases}0.4597701 & \text{for } N=30,\\ 0.4489338 & \text{for } N=100.\end{cases}$$

Since the 5%-quantiles of the full component 1-depth are 0.4344828 for $N=30$ and 0.4806061 for $N=100$ according to Table S.4 and the 2.5%-quantiles of the univariate full 1-simplex depth are 0.4344828 for $N=30$ and 0.4806061 for $N=100$ according to Table S.4, the null hypothesis would be never rejected by the second component for $N=30$ when the variance is so small so that the signs of the second component are always positive on $[-1,-1/3)$ and $(1/3,1]$ and always negative on $(-1/3,1/3)$. The rejection of the null hypothesis is then only possible with the first component and this probability is 0.025. Hence the power is approximately 0.025 for $N=30$. However, H_0 would be often rejected for $N=100$ by the second component so that the power would be close to 1.

The above behaviour is more pronounced when the univariate full 1-simplex depth (the full 2-sign depth) is only applied on the second component ignoring the first component. Then the 5%-quantiles should be used which are 0.4597701 for $N=30$ and 0.4848485 for $N=100$ according to Table S.4. Again we have almost no rejection for $N=30$ and almost always a rejection for $N=100$.

For the univariate full 2-simplex depth (the full 3-sign depth), the relative number of triples with alternating signs for the second component is

$$\frac{\left(\frac{N}{3}\right)^3}{\binom{N}{3}} \stackrel{N\to\infty}{\longrightarrow} \frac{6}{3^3}=\frac{2}{9}=0.2222222<\frac{1}{4}.$$

We also have

$$\frac{\left(\frac{N}{3}\right)^3}{\binom{N}{3}}=\begin{cases}0.2463054 & \text{for } N=30,\\ 0.2290478 & \text{for } N=100.\end{cases}$$

Since the 5%-quantiles of the full component 2-depth are 0.1931034 for $N=30$ and 0.2334694 for $N=100$ according to Table S.4 and the 2.5%-quantiles of the the univariate full 2-simplex depth are 0.1931034 for $N=30$ and 0.2333828 for $N=100$ according to Table S.4, the null hypothesis would be never rejected by the second component for $N=30$ when the variance is so small so that the signs of the second component are always positive on $[-1,-1/3)$ and $(1/3,1]$ and always negative on $(-1/3,1/3)$. The rejection of the null hypothesis is then only possible with the first component and this probability is 0.025. Hence, the power is low for $N = 30$. However, H_0 would be almost always rejected for $N = 100$ by the second component so that the power is quite good.

The above behaviour is more pronounced when the univariate full 2-simplex depth (the full 3-sign depth) is only applied on the second component ignoring the first component. Then the 5%-quantiles should be used which are 0.2068966 for $N = 30$ and 0.2376129 for $N = 100$ according to Table S.4. Then we have almost always no rejection for $N = 30$ and almost always a rejection for $N = 100$.

Quadratic regression: second model. Here we assume that the first $N/4$ signs of the second component $Y_{n,2}$ are positive, the second $N/2$ signs are negative, and the last $N/4$ are positive under the alternative.

For the bivariate full 1-simplex depth, only triples are counted with the sign constellations $++-$, $-++$, $+--$, $--+$, and $+-+$ of the second component. The relative number of these triples for each of the first two constellations is given by

$$\begin{aligned}
&\frac{1}{\binom{N}{3}}\binom{N/4}{2}\cdot\frac{N}{2}=\frac{1}{\binom{N}{3}}\frac{\frac{N}{4}\left(\frac{N}{4}-1\right)}{2}\cdot\frac{N}{2}\\
&=\frac{3\cdot 2\cdot N\,(N-4)\,N}{N\,(N-1)\,(N-2)\cdot 4\cdot 4\cdot 2\cdot 2}=\frac{N\,(N-4)\cdot 3}{(N-1)\cdot(N-2)\cdot 4\cdot 4\cdot 2}.
\end{aligned}$$

The relative number of the triples for the third and fourth constellation is given by

$$\begin{aligned}
&\frac{1}{\binom{N}{3}}\binom{N/2}{2}\cdot\frac{N}{4}=\frac{1}{\binom{N}{3}}\frac{\frac{N}{2}\left(\frac{N}{2}-1\right)}{2}\cdot\frac{N}{4}\\
&=\frac{3\cdot 2\cdot N\,(N-2)\,N}{N\,(N-1)\,(N-2)\cdot 2\cdot 2\cdot 2\cdot 4}=\frac{N\cdot 3}{(N-1)\cdot 4\cdot 4}.
\end{aligned}$$

The relative number of the triples for the last constellation is given by

$$\frac{1}{\binom{N}{3}}\left(\frac{N}{4}\right)^2\cdot\frac{N}{2}=\frac{3\cdot 2\cdot N^3}{N\,(N-1)\,(N-2)\cdot 4^2\cdot 2}=\frac{N^2\cdot 3}{(N-1)\,(N-2)\cdot 4^2}.$$

Hence the relative amount of triples where 0_2 can be included in the simplex is

$$\begin{aligned}
&2\,\frac{N\,(N-4)\cdot 3}{(N-1)\cdot(N-2)\cdot 4\cdot 4\cdot 2}+2\,\frac{N\cdot 3}{(N-1)\cdot 4\cdot 4}+\frac{N^2\cdot 3}{(N-1)\,(N-2)\cdot 4^2}\\
&=\frac{N\cdot 3}{(N-1)\,(N-2)\cdot 4^2}\left[N-4+2(N-2)+N\right]\\
&=\frac{N\cdot 3}{(N-1)\,(N-2)\cdot 4^2}\left[4N-8\right]=\frac{N\cdot 3}{(N-1)\,(N-2)\cdot 4}\left[N-2\right]
\end{aligned}$$

$$= \frac{N \cdot 3}{(N-1) \cdot 4}.$$

This quantity must be multiplied with the probability that appropriate first components $Y_{n,1}$ can be found which is given by Lemma S.5 (a) as $\frac{1}{3}$ so that we expect approximately

$$\frac{N \cdot 3}{(N-1) \cdot 4} \cdot \frac{1}{3} = \frac{N}{(N-1) \cdot 4} \overset{N \to \infty}{\longrightarrow} \frac{1}{4}$$

simplices containing 0_2 under the alternative where $\frac{1}{4}$ is the expected number also under H_0. We have

$$\frac{N}{(N-1) \cdot 4} = \begin{cases} 0.2586207 & \text{for } N = 30, \\ 0.2525253 & \text{for } N = 100. \end{cases}$$

This explains why we cannot reject H_0 with the bivariate full 1-simplex depth.

For the bivariate full 2-simplex depth, only quadruples can be counted with the sign constellations $++--$, $--++$, $++-+$, $+-++$, and $+--+$. The relative number of these quadruples of the first two constellations is given by

$$\begin{aligned} \frac{1}{\binom{N}{4}} \binom{N/4}{2} \cdot \binom{N/2}{2} &= \frac{1}{\binom{N}{4}} \frac{\frac{N}{4}\left(\frac{N}{4}-1\right)}{2} \cdot \frac{\frac{N}{2}\left(\frac{N}{2}-1\right)}{2} \\ &= \frac{4 \cdot 3 \cdot 2 \cdot N\,(N-4)\,N\,(N-2)}{N\,(N-1)\,(N-2)\,(N-3) \cdot 4^2 \cdot 2^2 \cdot 2^2} \\ &= \frac{3 \cdot (N-4)\,N}{(N-1)\,(N-3) \cdot 4^2 \cdot 2}. \end{aligned}$$

The relative number of the quadruples for the third and fourth constellations is given by

$$\begin{aligned} \frac{1}{\binom{N}{4}} \binom{N/4}{2} \cdot \frac{N}{2} \cdot \frac{N}{4} &= \frac{1}{\binom{N}{4}} \frac{\frac{N}{4}\left(\frac{N}{4}-1\right)}{2} \cdot \frac{N}{2} \cdot \frac{N}{4} \\ &= \frac{4 \cdot 3 \cdot 2 \cdot N\,(N-4)\,N^2}{N\,(N-1)\,(N-2)\,(N-3) \cdot 4^2 \cdot 4 \cdot 4} \\ &= \frac{3 \cdot (N-4)\,N^2}{(N-1)\,(N-2)\,(N-3) \cdot 4^2 \cdot 2}. \end{aligned}$$

The relative number of the quadruples for the last constellation is given by

$$\begin{aligned} \frac{1}{\binom{N}{4}} \frac{N}{4} \cdot \binom{N/2}{2} \cdot \frac{N}{4} &= \frac{1}{\binom{N}{4}} \frac{\frac{N}{2}\left(\frac{N}{2}-1\right)}{2} \cdot \frac{N^2}{4 \cdot 4} \\ &= \frac{4 \cdot 3 \cdot 2 \cdot N\,(N-2)\,N^2}{N\,(N-1)\,(N-2)\,(N-3) \cdot 4 \cdot 2 \cdot 4^2} \\ &= \frac{3 \cdot N^2}{(N-1)\,(N-3) \cdot 4^2}. \end{aligned}$$

These relative numbers must be multiplied with the probability that appropriate first components $Y_{n,1}$ can be found. This probability is $\frac{1}{12}$ for the first two constellations according to Lemma S.5 (d) and $\frac{1}{6}$ for the last three constellations according to Lemma S.5 (b) and (c). Hence we expect approximately

$$
\begin{aligned}
& 2 \cdot \frac{3 \cdot (N-4)\, N}{(N-1)\,(N-3) \cdot 4^2 \cdot 2} \cdot \frac{1}{12} + 2 \cdot \frac{3 \cdot (N-4)\, N^2}{(N-1)\,(N-2)\,(N-3) \cdot 4^2 \cdot 2} \cdot \frac{1}{6} \\
& +1 \cdot \frac{3 \cdot N^2}{(N-1)\,(N-3) \cdot 4^2} \cdot \frac{1}{6} \\
& = \frac{N}{(N-1)\,(N-2)\,(N-3) \cdot 4^2 \cdot 4}((N-2)(N-4) + 2\,(N-4)\,N \\
& \qquad\qquad + \ 2\,(N-2)\,N) \\
& = \frac{N}{(N-1)\,(N-2)\,(N-3) \cdot 4^2 \cdot 4} \\
& \qquad \cdot (N^2 - 4N - 2N + 8 + 2N^2 - 8N + 2N^2 - 4N) \\
& = \frac{N}{(N-1)\,(N-2)\,(N-3) \cdot 4^2 \cdot 4}\,(5N^2 - 18N + 8) \\
& \xrightarrow{N \to \infty} \frac{5}{4 \cdot 4 \cdot 4} = \frac{5}{64} = 0.078125 < \frac{1}{12} = 0.08333333
\end{aligned}
$$

pairs of simplices containing 0_2 under the alternative. We have

$$
\frac{N}{(N-1)\,(N-2)\,(N-3) \cdot 4^2 \cdot 4}\,(5N^2 - 18N + 8) = \begin{cases} 0.08483853 & \text{for } N = 30, \\ 0.08003983 & \text{for } N = 100. \end{cases}
$$

Note also that the 5%-quantiles are 0.06236088 and 0.07747859 for $N = 30$ and $N = 100$, respectively, according to Table S.2. Hence the chance for rejection of H_0 is low for $N = 30$ and $N = 100$ but should exist for larger N. However, it does not explain why the full 2-simplex depth appears to be worse than the full 1-simplex depth in Figure 5 in this model.

Note, that for the univariate full 1-simplex depth (the full 2-sign depth), the relative number of pairs with alternating signs for the second component is

$$
\frac{\frac{N}{4} \cdot \frac{N}{2} + \frac{N}{4} \cdot \frac{N}{2}}{\binom{N}{2}} \xrightarrow{N \to \infty} \frac{4}{8} = \frac{1}{2},
$$

and

$$
\frac{\frac{N}{4} \cdot \frac{N}{2} + \frac{N}{4} \cdot \frac{N}{2}}{\binom{N}{2}} = \begin{cases} 0.5172414 & \text{for } N = 30, \\ 0.5050505 & \text{for } N = 100. \end{cases}
$$

Since the 5%-quantiles of the full component 1-depth are 0.4344828 for $N = 30$ and 0.4806061 for $N = 100$ according to Table S.4 and the 2.5%-quantiles of the univariate full 1-simplex depth are 0.4344828 for $N = 30$ and 0.4806061 for $N = 100$ according to Table S.4, the null hypothesis would be never rejected by the second component for $N = 30$ as well as for $N = 100$. This also holds when the univariate full 1-simplex depth (the full 2-sign depth) is only applied on the second component ignoring the first component. Then the 5%-quantiles, which are 0.4597701 for $N = 30$ and 0.4848485 for $N = 100$ according to Table S.4, are still smaller than the relative number of pairs with alternating signs.

For the univariate full 2-simplex depth (the full 3-depth), the relative number of triples with alternating signs for the second component is

$$\frac{\left(\frac{N}{4}\right)^2 \cdot \frac{N}{2}}{\binom{N}{3}} \overset{N\to\infty}{\longrightarrow} \frac{6}{4 \cdot 4 \cdot 2} = \frac{3}{16} = 0.1875 < \frac{1}{4},$$

and

$$\frac{\left(\frac{N}{4}\right)^2 \cdot \frac{N}{2}}{\binom{N}{3}} = \begin{cases} 0.2078202 & \text{for } N = 30, \\ 0.1932591 & \text{for } N = 100. \end{cases}$$

Since the 5%-quantiles of the full component 2-depth are 0.1931034 for $N = 30$ and 0.2334694 for $N = 100$ according to Table S.4 and the 2.5%-quantiles of the the univariate full 2-simplex depth are 0.1931034 for $N = 30$ and 0.2333828 for $N = 100$ according to Table S.4, the null hypothesis would be never rejected by the second component for $N = 30$. The rejection of the null hypothesis is then only possible with the first component and this probability is 0.025. Hence, the power is low for $N = 30$. However, H_0 would be almost always rejected for $N = 100$ by the second component so that the power is high.

The above behaviour is more pronounced when the univariate full 2-simplex depth (the full 3-sign depth) is only applied on the second component ignoring the first component. Then the 5%-quantiles should be used which are 0.2068966 for $N = 30$ and 0.2376129 for $N = 100$ according to Table S.4. Then we have almost always no rejection for $N = 30$ and almost always a rejection for $N = 100$.

Since the relative number of triples with alternating signs for the second component is lower than in the first quadratic regression model, here, the chance of rejection is higher if the sign behaviour of the second component is not as strict as assumed.