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# Optimal designs for inspection times of interval-censored data

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**Abstract** We treat optimal equidistant and optimal non-equidistant inspection times for interval-censored data with exponential distribution. We provide in particular a recursive formula for calculating the optimal non-equidistant inspection times which is similar to a formula for optimal spacing of quantiles for asymptotically best linear estimates based on order statistics. This formula provides an upper bound for the standardized Fisher information which is reached for the optimal non-equidistant inspection times if the number of inspections is converging to infinity. The same upper bound is also shown for the optimal equidistant inspection times. Since optimal equidistant inspection times are easier to calculate and easier to handle in practice, we study the efficiency of optimal equidistant inspection times with respect to optimal non-equidistant inspection times. Moreover, since the optimal inspection times are only locally optimal, we provide also some results concerning maximin efficient designs.

**Keywords** Optimal inspection times · Exponential distribution · Optimal spacing of quantiles · Maximin efficient designs

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## 1 Introduction

Let $T_1, \ldots, T_N$ be independent nonnegative random variables (lifetime variables). However, the realizations $t_1, \ldots, t_N$ of $T_1, \ldots, T_N$ are not observed directly. Only realizations $n_i$ of

$$N_i := \sum_{n=1}^{N} \mathbb{1}_{(\tau_{i-1}, \tau_i]}(T_n), \quad i = 1, \ldots, I+1,$$

are observed, where $0 = \tau_0 < \tau_1 < \ldots < \tau_I < \tau_{I+1} = \infty$ are given inspection times and $\mathbb{1}_A$ denotes the indicator function for the set $A$. In particular, we have $N = \sum_{i=1}^{I+1} n_i$.

Such data are called interval-censored data or grouped data. They appear in particular in engineering science and medicine where failures of objects or diseases can only be detected at special inspection times. The analysis of such data is an old problem and was already treated in the book [11] from 1961. Nevertheless, it is still a very active research area. There are several new books on this topic as those of [25] and [4] and many recent papers as those of [3], [10], [2], [27], [7].

The question how to choose optimal inspection times $\tau_1 < \ldots < \tau_I$ was also treated already in the sixties of the last century. [11] listed locally optimal inspection times for exponential distribution for $I = 1, \ldots, 6$ and [15] extended these results for $I = 1, \ldots, 10$ for equally spaced inspections and optimally spaced inspections. [28], [29] provided tables for locally optimal inspection times for other distributions and [18], [8] studied a Bayesian approach to find optimal inspection times. After Aggarwala introduced progressive Type I interval censoring in 2001 ([1]), several papers as those of [13], [26], [31], [3] treated optimal inspection times for progressive interval censoring for several types of distributions. There are also other design considerations for interval-censored data, as the determination of the sample size for comparing several groups ([14]) or stress levels in accelerated life tests ([33], [23], [9], [32], [10]).

All these approaches did not provide much theory about the optimal inspection times. They only calculated the locally optimal inspection times numerically and provided then tables with the optimal inspection times.

However, as [20] and [17] noted, there is a relationship to optimal spacing of quantiles for asymptotically best linear estimates (ABLE) based on order statistics. The treatment of optimal spacing of quantiles started already in the forties of the last century (see [22], [21], [12], [6], [16]) and concerned several types of distributions. For the exponential distribution, Saleh provided a recursive formula for the optimal spacing in [21]. But this formula in Theorem 6.2 is not correct. It is probably a misprint of a formula in his Ph.D. thesis [20]. Although the optimal spacing of quantiles and the optimal non-equidistant inspection times are related, we are not aware that this wrong formula was corrected or used later for optimal inspection times.

In this paper, we provide a different recursive formula for optimal non-equidistant inspection times for exponential distribution from which the correct form of the formula of Saleh can easily be derived. Moreover, we use this formula to show that a standardized Fisher information is always less than 1 and is approaching 1 for the optimal inspection times if $I$ tends to infinity. We prove this upper bound not only for optimally spaced inspection times but also for optimally equidistantly spaced inspection times. This bound implies in particular that already $I = 5$ provides a high efficiency, similarly to a result found by numerical calculations in [24] for test procedures for exponential distribution and in [19] for the Weibull distribution.

In Section 2, the maximum likelihood estimator is presented and the corresponding Fisher information is given. Section 3 provides the optimal inspection times for the case of equidistantly spaced inspection times and Section 4 presents the results concerning the optimal non-equidistantly spaced inspection times. Since the optimal inspection times depend on the unknown parameter, i.e. they are only locally optimal, we discuss also maximin efficient designs in both sections. A comparison of locally optimal and maximin efficient equidistant and non-equidistant designs is given in Section 5. Finally, Section 6 provides a short discussion of the results.

## 2 Maximum likelihood estimator and the Fisher information

We assume that $T_n$ has an exponential distribution with unknown parameter $\lambda > 0$ and corresponding cumulative distribution function $F_\lambda$. Then the likelihood function for an observation $n_i$ is given by

$$\begin{aligned} l_\lambda(n_i) &:= \prod_{n=1}^{N} P_\lambda\left(T_n \in (\tau_{i-1}, \tau_i]\right)^{\mathbb{1}_{(\tau_{i-1},\tau_i]}(t_n)} \\ &= \left(F_\lambda(\tau_i) - F_\lambda(\tau_{i-1})\right)^{n_i} = \left(e^{-\lambda\tau_{i-1}} - e^{-\lambda\tau_i}\right)^{n_i} \end{aligned}$$

for $i = 1, \ldots, I$ and

$$l_\lambda(n_{I+1}) := \prod_{n=1}^{N} P_\lambda\left(T_n \in (\tau_I, \infty)\right)^{\mathbb{1}_{(\tau_I,\infty)}(t_n)} = \left(1 - F_\lambda(\tau_I)\right)^{n_{I+1}} = \left(e^{-\lambda\tau_I}\right)^{n_{I+1}}$$

so that the common likelihood function of $n_* := (n_1, \ldots, n_{I+1})$ is given by

$$L_\lambda(n_*) := \prod_{i=1}^{I} \left(e^{-\lambda\tau_{i-1}} - e^{-\lambda\tau_i}\right)^{n_i} \ \left(e^{-\lambda\tau_I}\right)^{n_{I+1}}. \tag{1}$$

The derivative of the loglikelihood function is then

$$\frac{\partial}{\partial\lambda} \ln L_\lambda(n_*) = \sum_{i=1}^{I} n_i \, \frac{\tau_i\, e^{-\lambda\tau_i} - \tau_{i-1}\, e^{-\lambda\tau_{i-1}}}{e^{-\lambda\tau_{i-1}} - e^{-\lambda\tau_i}} + n_{I+1}\,(-\tau_I) \tag{2}$$

so that a maximum likelihood estimator for $\lambda$ can be easily determined by maximizing (1) or calculating the root of (2).

To get the Fisher information, note at first that the likelihood function for a single random variable $T_n$, although $T_n$ is not observed, is

$$l_\lambda(T_n) := \prod_{i=1}^{I+1} P_\lambda\left(T_n \in (\tau_{i-1}, \tau_i]\right)^{\mathbb{1}_{(\tau_{i-1},\tau_i]}(T_n)}$$

so that

$$\frac{\partial}{\partial\lambda} \ln l_\lambda(T_n) = \sum_{i=1}^{I+1} \frac{\partial}{\partial\lambda} \ln\left(F_\lambda(\tau_i) - F_\lambda(\tau_{i-1})\right) \mathbb{1}_{(\tau_{i-1},\tau_i]}(T_n).$$

Hence, the Fisher information is given as

$$\begin{aligned}
&I_\lambda(\tau_1,\ldots,\tau_I) := E_\lambda\left(\left(\frac{\partial}{\partial\lambda}\ln l_\lambda(T_n)\right)^2\right)\\
&= \sum_{i=1}^{I+1} \frac{\left(\frac{\partial}{\partial\lambda}\left(F_\lambda(\tau_i) - F_\lambda(\tau_{i-1})\right)\right)^2}{\left(F_\lambda(\tau_i) - F_\lambda(\tau_{i-1})\right)^2}\left(F_\lambda(\tau_i) - F_\lambda(\tau_{i-1})\right)\\
&= \sum_{i=1}^{I+1} \frac{\left(\tau_i e^{-\lambda\tau_i} - \tau_{i-1}e^{-\lambda\tau_{i-1}}\right)^2}{e^{-\lambda\tau_{i-1}} - e^{-\lambda\tau_i}} = \frac{1}{\lambda^2}\left(\sum_{i=1}^{I+1} \frac{\left(\lambda\tau_i e^{-\lambda\tau_i} - \lambda\tau_{i-1}e^{-\lambda\tau_{i-1}}\right)^2}{e^{-\lambda\tau_{i-1}} - e^{-\lambda\tau_i}}\right)\\
&= \frac{1}{\lambda^2}\left(\sum_{i=1}^{I} \frac{\left(\lambda\tau_i e^{-\lambda\tau_i} - \lambda\tau_{i-1}e^{-\lambda\tau_{i-1}}\right)^2}{e^{-\lambda\tau_{i-1}} - e^{-\lambda\tau_i}} + (\lambda\tau_I)^2 e^{-\lambda\tau_I}\right). \qquad (3)
\end{aligned}$$

Thus, to find optimal inspection times $\tau_1^*,\ldots,\tau_I^*$ so that $I_\lambda(\tau_1,\ldots,\tau_I)$ is maximized, it is sufficient to use the substitution $x_i := \lambda\tau_i$ and to find $x_1^*,\ldots,x_I^*$ which maximize

$$f_I(x_1,\ldots,x_I) := \sum_{i=1}^{I} \frac{\left(x_i e^{-x_i} - x_{i-1}e^{-x_{i-1}}\right)^2}{e^{-x_{i-1}} - e^{-x_i}} + x_I^2 e^{-x_I}, \qquad (4)$$

where $x_0 = x_0^* = 0$. Thereby, $f_I(x_1,\ldots,x_I)$ is a standardized Fisher information. In particular, the optimal $x_1^*,\ldots,x_I^*$ satisfy that the quantity

$$\frac{1}{N\lambda^2} f_I(x_1^*,\ldots,x_I^*)^{-1}$$

is the asymptotic variance of the asymptotically best linear estimate (ABLE) for $\frac{1}{\lambda}$ based on order statistics for the exponential distribution and $x_1^*,\ldots,x_I^*$ are the quantiles of the so called optimal spacing of quantiles, see [22], [21]. These optimal quantiles have the advantage that they are independent of the unknown parameter $\lambda$ while the optimal inspection times depend on $\lambda$.

The following lemma provides a representation of $f_I$ in (4) which can be found in Theorem 6.2 in [21] in the context of optimal spacing of quantiles.

**Lemma 1** *The function $f_I(x_1,\ldots,x_I)$ in (4) can be simplified as follows*

$$f_I(x_1,\ldots,x_I) = \sum_{i=1}^{I} \frac{\left(x_i - x_{i-1}\right)^2}{e^{x_i} - e^{x_{i-1}}}. \qquad (5)$$

## 3 Optimal equidistant inspection times

At first, let us consider the special case of a design with equidistant inspection times $\tau_1 = \Delta, \tau_2 = 2\Delta, \ldots, \tau_I = I\Delta$. Equidistant designs are useful in applications because their implementation and realization is more convenient. In this case, (3) becomes

$$I_{\lambda,eq}(\Delta) = \frac{1}{\lambda^2}\left(\sum_{i=1}^{I} \frac{\left(\lambda i\Delta e^{-\lambda i\Delta} - \lambda(i-1)\Delta e^{-\lambda(i-1)\Delta}\right)^2}{e^{-\lambda(i-1)\Delta} - e^{-\lambda i\Delta}} + (\lambda I\Delta)^2 e^{-\lambda I\Delta}\right).$$

Again, with the substitution $x := \lambda\Delta$, the maximization of $I_{\lambda,eq}(\Delta)$ with respect to $\Delta$ is equivalent to the maximization of

$$f_{I,eq}(x) := \sum_{i=1}^{I} \frac{\left(i\, xe^{-i\,x} - (i-1)\, xe^{-(i-1)\,x}\right)^2}{e^{-(i-1)x} - e^{-i\,x}} + (I\,x)^2 e^{-I\,x}. \qquad (6)$$

Hence, the maximum $\Delta^*(\lambda) := \Delta^*(\lambda, I)$ of $I_{\lambda,eq}(\Delta)$ is given by $\Delta^*(\lambda) = \frac{x^*_{eq}}{\lambda}$ if $f_{I,eq}$ has a maximum at $x^*_{eq} := x^*_{eq}(I)$. The optimal equidistantly spaced inspection times are then $\Delta^*(\lambda), 2\Delta^*(\lambda), \ldots, I\Delta^*(\lambda)$.

**Lemma 2** *The function $f_{I,eq}(x)$ in (6) can be simplified as follows*

$$f_{I,eq}(x) = \frac{e^x x^2(1-e^{-Ix})}{(e^x-1)^2}, \ \textit{in particular}\ f_{1,eq}(x) = \frac{x^2}{e^x-1}. \qquad (7)$$

*Proof* Note that $f_{I,eq}(x)$ is a special case of the function $f_I(x_1, \ldots, x_I)$ from (4) with $x_i = ix$ for $i = 1, \ldots, I$. Lemma 1 yields then the assertion. □

The values $x^*_{eq}$ can be found numerically. Table 1 contains the first inspection point $x^*_{eq}$, the last inspection point $Ix^*_{eq}$ and the maximum of the function $f_{I,eq}$ for some values of $I$. Moreover, Figure 1 shows the functions $f_{I,eq}$ for $I = 1, 5, 10, 20, 50$ and the corresponding maximum points.

**Table 1** Equidistant case: $x^*_{eq}$, $Ix^*_{eq}$ and $f_{I,eq}(x^*_{eq})$

| $I$ | $x^*_{eq}$ | $Ix^*_{eq}$ | $f_{I,eq}(x^*_{eq})$ |
|---|---|---|---|
| 1 | 1.5936 | 1.5936 | 0.6476 |
| 5 | 0.7456 | 3.7280 | 0.9320 |
| 10 | 0.4833 | 4.8330 | 0.9730 |
| 15 | 0.3672 | 5.5080 | 0.9848 |
| 20 | 0.2998 | 5.9960 | 0.9901 |
| 25 | 0.2552 | 6.3800 | 0.9929 |
| 30 | 0.2232 | 6.6960 | 0.9946 |
| 40 | 0.1799 | 7.1960 | 0.9966 |
| 50 | 0.1518 | 7.5900 | 0.9976 |

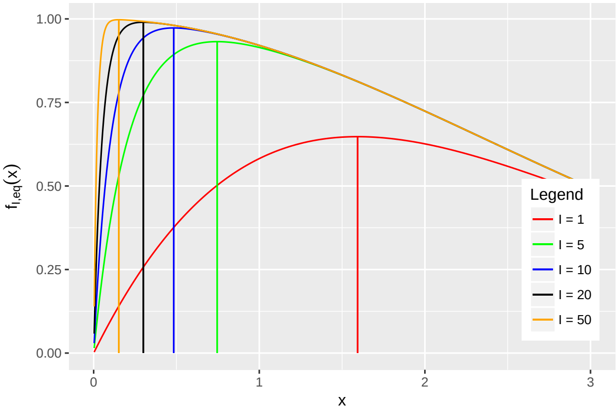


**Fig. 1** The functions $f_{I,eq}$ for $I = 1, 5, 10, 20, 50$ and the corresponding maximum points

**Theorem 1** *For the function $f_{I,eq}(x)$ in (6), the following holds:*

*(i)* $f_{I,eq}(x) \leq 1$ *for all* $I \in \mathbb{N}$ *and* $x > 0$.

*(ii)* $\max\{f_{I,eq}(x);\ x > 0\} \to 1$ *as* $I \to \infty$.

*(iii)* $f_{I,eq}$ *is a unimodal function for each* $I \in \mathbb{N}$.

*Proof* **(i)** Since $f_{I,eq}(x)$ is a special case of the function $f_I(x_1, \ldots, x_I)$ from (4) with $x_i = ix$ for $i = 1, \ldots, I$, the statement (i) follows from Theorem 2 (i) in Section 4. Hence, we show here only (ii) and (iii).
**(ii)** Since $1 - e^{-I_1 x} < 1 - e^{-I_2 x}$ for $I_1 < I_2$ and $x > 0$, we have

$$\max\{f_{I_1,eq}(x);\ x > 0\} < \max\{f_{I_2,eq}(x);\ x > 0\}$$

so that $a_I := \max\{f_{I,eq}(x);\ x > 0\}$, $I \in \mathbb{N}$, is an increasing sequence. From (i) it follows that $a_I \leq 1$ for all $I \in \mathbb{N}$. This yields

$$\lim_{I \to \infty} a_I = a_\infty \leq 1.$$

Consider the function $f_{I,eq}(x)$ with $x = 1/\sqrt{I}$:

$$f_{I,eq}\left(1/\sqrt{I}\right) = \frac{e^{1/\sqrt{I}}(1 - e^{-\sqrt{I}})}{I\,(e^{1/\sqrt{I}} - 1)^2}.$$

Using the substitution $y := 1/\sqrt{I}$ and L'Hospital's rule, we obtain

$$\lim_{I \to \infty} f_{I,eq}\left(1/\sqrt{I}\right) = \lim_{y \to 0} \frac{e^y y^2 (1 - e^{-1/y})}{(e^y - 1)^2} = 1.$$

Since $f_{I,eq}\left(1/\sqrt{I}\right) \leq a_I$ by definition, we obtain

$$a_\infty \geq \lim_{I\to\infty} f_{I,eq}\left(1/\sqrt{I}\right) = 1.$$

Therefore, $a_\infty = 1$.

**(iii)** For unimodality it is sufficient to show that $f_{I,eq}$ has only one extremum and this extremum is maximum point. The first derivative of $f_{I,eq}$ is

$$f'_{I,eq}(x) = \frac{xe^x\left(2e^x - 2 - x - xe^x + e^{-Ix}(2 + x + xe^x - 2e^x + Ix(e^x-1))\right)}{(e^x-1)^3}.$$

Define $q(x) := 2 + x + xe^x - 2e^x$ for $x \geq 0$. Note that $q(0) = 0$ and that $q(x)$ is strictly increasing since $q'(x) = 1 + e^x(x-1)$ and $e^x < 1/(1-x)$ for $0 < x < 1$. So, $q'(x) > 0$ for all $x > 0$. Therefore, $q(x) > 0$ for $x > 0$. Using this fact, we rewrite $f'_{I,eq}(x)$ as follows:

$$f'_{I,eq}(x) = \frac{xe^x\, q(x)\, e^{-Ix}(Ix)}{(e^x-1)^3}\left(\frac{1-e^{Ix}}{Ix} + \frac{1}{\frac{x(e^x+1)}{e^x-1} - 2}\right).$$

Since $x > 0$, $f'_{I,eq}(x) = 0$ is equivalent to

$$p(x) := \frac{1-e^{Ix}}{Ix} + \frac{1}{\frac{x(e^x+1)}{e^x-1} - 2} = 0.$$

Note that the function $\frac{1-e^{Ix}}{Ix}$ is decreasing for $x > 0$:

$$\frac{d}{dx}\left(\frac{1-e^{Ix}}{Ix}\right) = \frac{e^{Ix}(1-Ix)-1}{Ix^2} < 0, \quad \text{since } e^x < \frac{1}{1-x} \text{ for } 0 < x < 1.$$

Since $e^x > 1 + x + x^2/2$ and, consequently, $(e^x - x)^2 > (1 + x^2/2)^2$ for $x > 0$, we show that $\frac{x(e^x+1)}{e^x-1}$ is increasing for $x > 0$:

$$\frac{d}{dx}\left(\frac{x(e^x+1)}{e^x-1}\right) = \frac{e^{2x} - 2xe^x - 1}{(e^x-1)^2} = \frac{(e^x-x)^2 - x^2 - 1}{(e^x-1)^2} > \frac{x^4}{4(e^x-1)^2} > 0.$$

This makes the function $p(x)$ decreasing for $x > 0$. Moreover, it is easy to check that the function $p(x)$ is a continuous function with $\lim_{x\to 0} p(x) = +\infty$ and $\lim_{x\to+\infty} p(x) = -\infty$. Hence, there exists only one $x_0 > 0$ such that $p(x_0) = 0$ with $p(x) < 0$ for $x > x_0$ and $p(x) > 0$ for $x < x_0$. $\square$

*Remark 1* From Theorem 1 and from Table 1, it follows that already with $I = 5$ equidistant inspections we obtain more than 93% of the maximum information. Note that the maximum information coincides with the information of the maximum likelihood estimator for non-censored lifetimes.

The efficiency of a given equidistant partition $\Delta, 2\Delta, \dots, I\Delta$ with respect to the locally optimal equidistantly spaced inspections $\Delta^*(\lambda), 2\Delta^*(\lambda), \dots, I\Delta^*(\lambda)$ with $\Delta^*(\lambda) = x^*_{eq}/\lambda$ is given by

$$\frac{I_{\lambda,eq}(\Delta)}{I_{\lambda,eq}(\Delta^*(\lambda))} = \frac{\frac{1}{\lambda^2} f_{I,eq}(\lambda\Delta)}{\frac{1}{\lambda^2} f_{I,eq}(x^*_{eq})} = \frac{f_{I,eq}(\lambda\Delta)}{f_{I,eq}(x^*_{eq})}.$$

Lemma 2 yields

$$\frac{I_{\lambda,eq}(\Delta)}{I_{\lambda,eq}(\Delta^*(\lambda))} = \frac{e^{\lambda\Delta}(\lambda\Delta)^2(1-e^{-I\lambda\Delta})}{f_{I,eq}(x^*_{eq})(e^{\lambda\Delta}-1)^2}.$$

Since $\Delta > 0$, it follows

$$\lim_{\lambda\to\infty} \frac{e^{\lambda\Delta}(\lambda\Delta)^2(1-e^{-I\lambda\Delta})}{(e^{\lambda\Delta}-1)^2} = \lim_{\lambda\to\infty} \frac{\frac{(\lambda\Delta)^2}{e^{\lambda\Delta}}(1-e^{-I\lambda\Delta})}{(1-\frac{1}{e^{\lambda\Delta}})^2} = 0.$$

Using L'Hospital's rule, we obtain

$$\lim_{\lambda\to 0} \frac{e^{\lambda\Delta}(\lambda\Delta)^2(1-e^{-I\lambda\Delta})}{(e^{\lambda\Delta}-1)^2} = \lim_{x\to 0} \frac{e^x x^2(1-e^{-Ix})}{(e^x-1)^2} = 0.$$

Hence, we have

$$\lim_{\lambda\to 0} \frac{I_{\lambda,eq}(\Delta)}{I_{\lambda,eq}(\Delta^*(\lambda))} = 0 = \lim_{\lambda\to\infty} \frac{I_{\lambda,eq}(\Delta)}{I_{\lambda,eq}(\Delta^*(\lambda))}$$

so that $\lambda$ must be restricted by a lower bound $L$ and an upper bound $U$ to get maximin efficient inspection times.
Since

$$f_{I,eq}(x) = \frac{e^x x^2(1-e^{-Ix})}{(e^x-1)^2}$$

is a unimodal function for each $I \in \mathbb{N}$ (see Theorem 1), a maximin efficient inspection distance $\Delta^*_{L,U}$ for $\lambda \in [L,U]$ is defined by

$$\begin{aligned}\Delta^*_{L,U} &:= \Delta^*([L,U]) := \arg\max_{\Delta>0} \min_{\lambda\in[L,U]} \frac{I_{\lambda,eq}(\Delta)}{I_{\lambda,eq}(\Delta^*(\lambda))} \\ &= \arg\max_{\Delta>0} \min\left\{\frac{e^{L\Delta}(L\Delta)^2(1-e^{-IL\Delta})}{(e^{L\Delta}-1)^2}, \frac{e^{U\Delta}(U\Delta)^2(1-e^{-IU\Delta})}{(e^{U\Delta}-1)^2}\right\} \frac{1}{f_{I,eq}(x^*_{eq})}.\end{aligned}$$

This means that the maximin efficient $\Delta^*_{L,U}$ must satisfy

$$\frac{e^{L\Delta^*_{L,U}}(L\Delta^*_{L,U})^2(1-e^{-IL\Delta^*_{L,U}})}{(e^{L\Delta^*_{L,U}}-1)^2} = \frac{e^{U\Delta^*_{L,U}}(U\Delta^*_{L,U})^2(1-e^{-IU\Delta^*_{L,U}})}{(e^{U\Delta^*_{L,U}}-1)^2},$$

or equivalently

$$f_{I,eq}(L\Delta^*_{L,U}) = f_{I,eq}(U\Delta^*_{L,U}).$$

Hence, the following lemma is obvious.

**Lemma 3** *If $\Delta^*_{L,U}$ is maximin efficient for $\lambda \in [L,U]$ then $\alpha\Delta^*_{L,U}$ is maximin efficient for $\lambda \in \left[\frac{L}{\alpha}, \frac{U}{\alpha}\right]$ for any $\alpha > 0$.*

## 4 Optimal non-equidistant inspection times

The aim of this section is to determine an optimal choice of the inspection times $\tau_1, \ldots, \tau_I$ for a fixed number $I$ of inspections. We want to find $\tau_1^*(\lambda) := \tau_{1,I}^*(\lambda), \ldots, \tau_I^*(\lambda) := \tau_{I,I}^*(\lambda)$ so that the information $I_\lambda(\tau_1, \ldots, \tau_I)$ in (3) is maximized. According to Section 2, it is sufficient to find $x_1^* := x_{1,I}^*$, …, $x_I^* := x_{I,I}^*$ which maximize $f_I(x_1, \ldots, x_I)$ given by (4) or (5). Then (3) is maximized by $\tau_1^*(\lambda) = \frac{x_1^*}{\lambda}, \ldots, \tau_I^*(\lambda) = \frac{x_I^*}{\lambda}$, where $x_1^*, \ldots, x_I^*$ can be determined numerically. For the optimal spacing of quantiles of asymptotically best linear estimates based on order statistics, this was done already in [22] for $I = 1, \ldots, 15$. For optimal inspection times, this was done in [11] for $I = 1, \ldots, 6$ and in [15] for $I = 1, \ldots, 10$. Table 2 provides some values for $I$ up to 50 which were calculated with Wolfram Mathematica [30].

**Table 2** Optimal values $x_1^*, x_2^*, \ldots, x_I^*$ and $f_I(x_1^*, ..., x_I^*)$

| $I$ | $x_1^*$ | $x_2^*$ | $x_3^*$ | ... | $x_{I-2}^*$ | $x_{I-1}^*$ | $x_I^*$ | $f_I(x_1^*, ..., x_I^*)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.594 | | | | | | | 0.6476 |
| 5 | 0.499 | 1.100 | 1.854 | | 1.854 | 2.871 | 4.465 | 0.9476 |
| 10 | 0.272 | 0.571 | 0.903 | ... | 3.559 | 4.576 | 6.170 | 0.9832 |
| 15 | 0.187 | 0.386 | 0.600 | ... | 4.638 | 5.655 | 7.249 | 0.9918 |
| 20 | 0.143 | 0.292 | 0.450 | ... | 5.430 | 6.447 | 8.041 | 0.9952 |
| 25 | 0.115 | 0.235 | 0.360 | ... | 6.056 | 7.073 | 8.667 | 0.9968 |
| 30 | 0.097 | 0.196 | 0.300 | ... | 6.573 | 7.590 | 9.184 | 0.9977 |
| 40 | 0.073 | 0.148 | 0.225 | ... | 7.398 | 8.415 | 10.007 | 0.9987 |
| 50 | 0.059 | 0.119 | 0.180 | ... | 8.046 | 9.063 | 10.657 | 0.9992 |

After analyzing the values in Table 2, we notice that the distances between the last $x_I^*$ and the second last $x_{I-1}^*$ are the same for all $I \in \mathbb{N}$. The same holds for other distances $d_i^* := d_{i,I}^* = x_i^* - x_{i-1}^*$, $i = 1, \ldots, I$ (see Table 3).

**Table 3** Optimal distances $d_1^*, d_2^*, \ldots, d_I^*$

| $I$ | $d_1^*$ | $d_2^*$ | $d_3^*$ | ... | $d_{I-2}^*$ | $d_{I-1}^*$ | $d_I^*$ |
|---|---|---|---|---|---|---|---|
| 1 | 1.594 | | | | | | |
| 5 | 0.499 | 0.601 | 0.754 | | 0.754 | 1.017 | 1.594 |
| 10 | 0.272 | 0.299 | 0.332 | ... | 0.754 | 1.017 | 1.594 |
| 15 | 0.187 | 0.199 | 0.214 | ... | 0.754 | 1.017 | 1.594 |
| 20 | 0.143 | 0.149 | 0.158 | ... | 0.754 | 1.017 | 1.594 |
| 25 | 0.115 | 0.120 | 0.125 | ... | 0.754 | 1.017 | 1.594 |
| ... | | | ... | | | | ... |

The expression for $f_I$ in (5) was maximized in Theorem 6.2 in [21]. In [21], the recursion

$$x_{i+1,I}^* = x_{i,I-1}^* + x_{i,I}^*, \quad i = 1, \ldots, I-1, \tag{8}$$

was given where $x^*_{1,I-1}, \dots, x^*_{I-1,I-1}$ and $x^*_{1,I}, \dots, x^*_{I,I}$ are the solutions for $I-1$ and $I$, respectively. However, this seems to be a misprint when comparing it with the Ph.D. thesis [20] of Saleh, where he proved in Theorem 4.2

$$x^*_{i+1,I} = x^*_{i,I-1} + x^*_{1,I}, \quad i = 1, \dots, I-1.$$

According to our observation that the last distances are always the same, (8) can be corrected alternatively to

$$x^*_{i+1,I} = x^*_{i,I-1} - x^*_{i-1,I-1} + x^*_{i,I}, \quad i = 1, \dots, I-1.$$

This follows immediately from the following theorem.

**Theorem 2** *For the function $f_I(x_1, \dots, x_I)$ in (4), the following holds:*

*(i)* $f_I(x_1, \dots, x_I) \leq 1 \qquad$ *for all* $\qquad I \in \mathbb{N} \qquad$ *and* $\qquad x_1, ..., x_I > 0.$

*(ii) Let*

$$(x^*_{1,I}, \dots, x^*_{I,I}) := \arg\max\{f_I(x_1, \dots, x_I);\ x_1, ..., x_I > 0\}$$

*and*

$$d^*_{i,I} := x^*_{i,I} - x^*_{i-1,I}, \qquad i = 1, \dots, I.$$

*Then the distances $d^*_{i,I}$ have the following property:*

$$d^*_{I_1-k,I_1} = d^*_{I_2-k,I_2} \qquad \text{for all} \quad I_1, I_2 \in \mathbb{N}, \quad k = 0, ..., \min\{I_1, I_2\} - 1.$$

*(iii)* $\max\{f_I(x_1, \dots, x_I);\ x_1, ..., x_I > 0\} \to 1 \qquad$ *as* $\qquad I \to \infty.$

*Proof* **(i)** Let $d_i := x_i - x_{i-1}$ for $i = 1, \dots, I$, where $x_0 := d_0 := 0$. Then (5) yields

$$f_I(x_1, ..., x_I) = \sum_{i=1}^{I} \frac{d_i^2}{e^{x_{i-1}}(e^{d_i} - 1)} = \sum_{i=1}^{I} \frac{d_i^2}{e^{d_1 + \dots + d_{i-1}}(e^{d_i} - 1)} =: \tilde{f}_I(d_1, ..., d_I).$$

Notice that $\tilde{f}_I(d_1, \dots, d_I)$ can be represented as

$$\begin{aligned} \tilde{f}_I(d_1, \dots, d_I) &= \frac{d_1^2}{e^{d_1} - 1} + \frac{1}{e^{d_1}} \left( \frac{d_2^2}{e^{d_2} - 1} + \frac{1}{e^{d_2}} \left( \frac{d_3^2}{e^{d_3} - 1} + \frac{1}{e^{d_3}} \left( \quad ... \right.\right.\right. \\ &\quad \left.\left.\left. ... \left( \frac{d_{I-1}^2}{e^{d_{I-1}} - 1} + \frac{1}{e^{d_{I-1}}} \left( \frac{d_I^2}{e^{d_I} - 1} \right) \right) ... \right) \right) \right) \\ &= g(d_1, g(d_2, g(d_3, g(...g(d_{I-1}, g(d_I, 0))...)))), \end{aligned} \tag{9}$$

where

$$g(t, c) := \frac{t^2}{e^t - 1} + \frac{1}{e^t}\, c, \qquad t \geq 0,\ c \geq 0.$$

Note that $g(t,0) = f_{1,eq}(t)$ in (7). The function $g(t,c)$ is increasing with respect to $c$ for each $t \geq 0$. In particular,

$$g(t,c) \leq g(t,1) \qquad \text{for} \qquad c \leq 1.$$

Note that $g(t,1)$ is decreasing function with respect to $t$:

$$\frac{d}{dt} g(t,1) = -\frac{(e^t(t-1)+1)^2}{e^t(e^t-1)^2} < 0,$$

and reaches its maximum at $t = 0$, so that $g(t,1) \leq g(0,1) = 1$. Therefore, for all $t \geq 0$ it holds

$$g(t,c) \leq 1 \qquad \text{for} \qquad c \leq 1,$$

which implies that $g(d_I, 0) < 1$ in (9) and, consequently, $\tilde{f}_I(d_1,\ldots,d_I) \leq 1$ for all $d_1,\ldots,d_I > 0$.

**(ii)** Consider the representation (9) of $f_I(x_1,\ldots,x_I)$. Since $g(t,c)$ is an increasing function with respect to $c$ for each $t \geq 0$, it holds:

The function $\tilde{f}_I(d_1,\ldots,d_I)$ is maximized at

$$\begin{aligned}
d_I^* &:= d_{I,I}^* = \arg\max\{g(t,0);\ t > 0\},\\
d_{I-1}^* &:= d_{I-1,I}^* = \arg\max\{g(t,g(d_I^*,0));\ t > 0\}, \qquad (10)\\
d_i^* &:= d_{i,I}^* = \arg\max\{g(t,g(d_{i+1}^*,g(...g(d_I^*,0)...)));\ t > 0\}, \quad i = 1,...,I-2.
\end{aligned}$$

Hence, the last optimal distance $d_{I,I}^*$ does not depend on $I$ and can be found numerically: $d_{I,I}^* \approx 1.594$ for all $I \in \mathbb{N}$ (see Figure 1 or Tables 1, 2, 3 for the case $I = 1$). The same holds for the second last distance $d_{I-1,I}^* \approx \arg\max\{g(t,0.6476);\ t > 0\} \approx 1.017$ (see Table 3) and so on. Hence, for all $I_1, I_2 \in \mathbb{N}$, we have

$$d_{I_1-k,I_1}^* = d_{I_2-k,I_2}^*, \qquad k = 0,...,\min\{I_1,I_2\}-1.$$

**(iii)** We divide the proof into two parts. In the first step we show that $\max\{f_I(x_1,\ldots,x_I);\ x_1,...,x_I > 0\} \to c_\infty \leq 1$ as $I \to \infty$ and in the second step we prove that $c_\infty = 1$.

*Step 1.* Note that representation (9) yields

$$\max\{f_I(x_1,\ldots,x_I);\ x_1,...,x_I > 0\} = \max\{\tilde{f}_I(d_1,\ldots,d_I);\ d_1,...,d_I > 0\}.$$

Let us show that $\max\{\tilde{f}_I(d_1,\ldots,d_I);\ d_1,...,d_I > 0\} \to c_\infty \leq 1$ as $I \to \infty$. It follows from (ii) that

$$\begin{aligned}
\max\{\tilde{f}_I(d_1,\ldots,d_I);\ d_1,...,d_I > 0\} &= \tilde{f}_I(d_1^*,\ldots,d_I^*)\\
&= g(d_1^*, g(d_2^*, g(d_3^*, g(...g(d_{I-1}^*, g(d_I^*,0))...)))),
\end{aligned}$$

where $d_1^*,\ldots,d_I^*$ are given by (10).

Let $c_0, c_1, c_2, \ldots$ be defined inductively via

$$c_i := \max\{g(t, c_{i-1});\ t \geq 0\}, \quad i = 1, \ldots, I, \quad c_0 := 0. \tag{11}$$

Note that (10) implies

$$g(d_I^*, 0) = c_1, \quad g(d_{I-1}^*, g(d_I^*, 0)) = c_2, \ldots,$$
$$g(d_i^*, g(...g(d_{I-1}^*, g(d_I^*, 0))...) = c_{I+1-i}, \quad i = 1, ..., I-2.$$

Hence, $\max\{\tilde{f}_I(d_1, \ldots, d_I);\ d_1, ..., d_I > 0\} = \tilde{f}_I(d_1^*, \ldots, d_I^*) = c_I$. Therefore, it is sufficient to show that $c_i \to c_\infty \leq 1$ as $i \to \infty$. Since the function $g(t, c)$ is increasing with respect to $c$ for each $t \geq 0$, we obtain

$$c' < c'' \quad \Longrightarrow \quad \max\{g(t, c');\ t \geq 0\} < \max\{g(t, c'');\ t \geq 0\}. \tag{12}$$

Note that (12) and the recursive definition (11) imply by induction that $(c_i)_{i \geq 1}$ is an increasing sequence provided we establish the induction basis $c_0 < c_1$. This base case can be shown numerically (see Figure 1 or Tables 1, 2 for $I = 1$):

$$c_1 = \max\{g(t, 0);\, t \geq 0\} \approx 0.6476 > 0 = c_0.$$

In (i) we showed that $g(t, c) \leq 1$ for $c \leq 1$. Therefore, $c_i \leq 1$ for all $i \in \mathbb{N}$. This means that the sequence $c_0, c_1, \ldots$ is an increasing sequence, which is bounded by 1. Hence, $c_i \to c_\infty \leq 1$ as $i \to \infty$.

*Step 2.* Define $h(c) := \max\{g(t, c);\ t \geq 0\}$ for $c \geq 0$. We will prove that $c_\infty = 1$ by showing the following: (a) $c_\infty = h(c_\infty)$; (b) $c < h(c)$ for $c \in (0, 1)$.
(a) At first let us show that $h$ is a continuous function on $[0, \infty)$. By definition, we have to show: for any $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $c', c''$ with $|c' - c''| \leq \delta$, the following holds

$$|h(c') - h(c'')| \leq \varepsilon.$$

By symmetry, we may assume that $c' \geq c''$. Let $t' := \arg\max\{g(t, c');\ t \geq 0\}$, $t'' := \arg\max\{g(t, c'');\ t \geq 0\}$ and $\delta := \varepsilon$. Then using the fact that $g(t, c)$ and, consequently, $h(c)$ is non-decreasing in $c$, we obtain:

$$\begin{aligned} &|h(c') - h(c'')| = h(c') - h(c'') = g(t', c') - g(t'', c'') \\ &\leq g(t', c') - g(t', c'') = \frac{1}{e^{t'}}(c' - c'') \leq \varepsilon. \end{aligned}$$

Thus, $h$ is continuous. In Step 1 we showed that $c_i \to c_\infty$ as $i \to \infty$, where $c_i = h(c_{i-1})$ with $c_0 = 0$. The continuity of $h$ yields $c_\infty = h(c_\infty)$.
(b) Let us show that $c < h(c)$ for $c \in (0, 1)$. Consider

$$g(t, c) - c = \frac{t^2}{e^t - 1} + \frac{1}{e^t}\, c - c = \frac{t^2}{e^t - 1} + \frac{1 - e^t}{e^t}\, c = \frac{t^2 e^t - c(e^t - 1)^2}{e^t(e^t - 1)}.$$

The fact that $e^t \leq 1/(1-t)$ for any $t \in [0, 1)$ yields $(e^t - 1)^2 \leq (t/(1-t))^2$ and

$$g(t, c) - c \geq \frac{t^2 e^t - \frac{ct^2}{(1-t)^2}}{e^t(e^t - 1)} = \frac{t^2\left(e^t - \frac{c}{(1-t)^2}\right)}{e^t(e^t - 1)}$$

for $t \in (0,1)$. Let $t_0 = 1-\sqrt{c}$. Note that $t_0 \in (0,1)$, since $c \in (0,1)$. Then

$$g(t_0,c) - c \geq \frac{(1-\sqrt{c})^2\left(e^{1-\sqrt{c}}-1\right)}{e^{1-\sqrt{c}}(e^{1-\sqrt{c}}-1)} > 0$$

and, consequently, $h(c) \geq g(t_0,c) > c$ for all $c \in (0,1)$.
Suppose that $c_\infty < 1$. Then it follows that $c_\infty < h(c_\infty)$ which contradicts the fact that $c_\infty = h(c_\infty)$ (see above). So, $c_\infty = 1$. □

*Remark 2* From Theorem 2 and from Tables 2, it follows that already with $I = 5$ inspections we obtain more than 94% of the maximum information.

The efficiency of a given partition $\tau_1, \ldots, \tau_I$ with respect to the locally optimal inspections $\tau_1^*(\lambda) = \frac{x_1^*}{\lambda}, \ldots, \tau_I^*(\lambda) = \frac{x_I^*}{\lambda}$ is given by

$$\begin{aligned} &\frac{I_\lambda(\tau_1,\ldots,\tau_I)}{I_\lambda(\tau_1^*(\lambda),\ldots,\tau_I^*(\lambda))} \qquad (13)\\ &= \frac{\frac{1}{\lambda^2}\sum_{i=1}^{I+1}\frac{\left(\lambda\tau_i e^{-\lambda\tau_i}-\lambda\tau_{i-1}e^{-\lambda\tau_{i-1}}\right)^2}{e^{-\lambda\tau_{i-1}}-e^{-\lambda\tau_i}}}{\frac{1}{\lambda^2}f_I(x_1^*,\ldots,x_I^*)} = \frac{\sum_{i=1}^{I+1}\frac{\left(\lambda\tau_i e^{-\lambda\tau_i}-\lambda\tau_{i-1}e^{-\lambda\tau_{i-1}}\right)^2}{e^{-\lambda\tau_{i-1}}-e^{-\lambda\tau_i}}}{f_I(x_1^*,\ldots,x_I^*)}. \end{aligned}$$

In particular, we have for $i = 1, \ldots, I+1$

$$\begin{aligned} &\lim_{\lambda\to\infty}\frac{\left(\lambda\tau_i e^{-\lambda\tau_i}-\lambda\tau_{i-1}e^{-\lambda\tau_{i-1}}\right)^2}{e^{-\lambda\tau_{i-1}}-e^{-\lambda\tau_i}}\\ &= \lim_{\lambda\to\infty}\frac{e^{-\lambda\tau_{i-1}}\left(\lambda\tau_i e^{-\lambda(\tau_i-\tau_{i-1})}-\lambda\tau_{i-1}\right)^2}{1-e^{-\lambda(\tau_i-\tau_{i-1})}} = 0. \end{aligned}$$

Also, using the L'Hospital's rule, we obtain for $i = 1, \ldots, I+1$

$$\lim_{\lambda\to 0}\frac{e^{-\lambda\tau_{i-1}}\left(\lambda\tau_i e^{-\lambda(\tau_i-\tau_{i-1})}-\lambda\tau_{i-1}\right)^2}{1-e^{-\lambda(\tau_i-\tau_{i-1})}} = 0.$$

Hence, we have again

$$\lim_{\lambda\to 0}\frac{I_\lambda(\tau_1,\ldots,\tau_I)}{I_\lambda(\tau_1^*(\lambda),\ldots,\tau_I^*(\lambda))} = 0 = \lim_{\lambda\to\infty}\frac{I_\lambda(\tau_1,\ldots,\tau_I)}{I_\lambda(\tau_1^*(\lambda),\ldots,\tau_I^*(\lambda))}$$

so that $\lambda$ must be restricted by a lower bound $L$ and an upper bound $U$ to get maximin efficient inspection times $\boldsymbol{\tau}_{L,U}^* := (\tau_1^*([L,U]),\ldots,\tau_I^*([L,U]))$ defined by

$$\boldsymbol{\tau}_{L,U}^* := \arg\max_{(\tau_1,\ldots,\tau_I)\in(0,\infty)^I}\ \min_{\lambda\in[L,U]}\frac{I_\lambda(\tau_1,\ldots,\tau_I)}{I_\lambda(\tau_1^*(\lambda),\ldots,\tau_I^*(\lambda))}.$$

Analogously to Lemma 3 we have with (13) the following lemma.

**Lemma 4** *If $\boldsymbol{\tau}_{L,U}^*$ is maximin efficient for $\lambda \in [L,U]$ then $\alpha\boldsymbol{\tau}_{L,U}^*$ is maximin efficient for $\lambda \in \left[\frac{L}{\alpha},\frac{U}{\alpha}\right]$ for any $\alpha > 0$.*

## 5 Comparison of the optimal and optimal equidistantly spaced inspection times

Let us compare the equidistant and the non-equidistant cases. In Figure 2, we see how the design points are spread and how fast the maxima of the functions $f_I$ and $f_{I,eq}$ converge to 1.

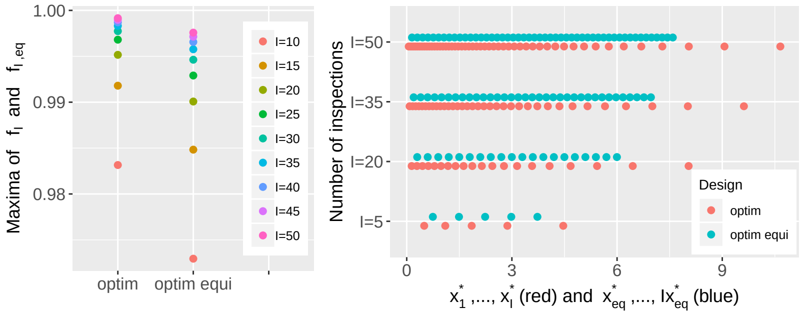


**Fig. 2** Maximum points of the functions $f_I$ and $f_{I,eq}$ (on the left) and the optimal $(x_1^*, \ldots, x_I^*)$ and the optimal equidistant $(x_{eq}^*, \ldots, Ix_{eq}^*)$ (on the right) for some values of $I$

Let us calculate the efficiency of the locally optimal equidistantly spaced inspections $\Delta^*(\lambda), 2\Delta^*(\lambda), \ldots, I\Delta^*(\lambda)$ with respect to the locally optimal non-equidistant inspections $\tau_1^*(\lambda), \ldots, \tau_I^*(\lambda)$. Sections 3 and 4 yield

$$\frac{I_\lambda(\Delta^*(\lambda), \ldots, I\Delta^*(\lambda))}{I_\lambda(\tau_1^*(\lambda), \ldots, \tau_I^*(\lambda))} = \frac{f_{I,eq}(x_{eq}^*)}{f_I(x_1^*, \ldots, x_I^*)} =: g(I),$$

i.e. the efficiency does not depend on parameter $\lambda$. Table 4 provides the efficiency of the equidistant design for some values of $I$. We see that the equidistant design yields nearly the same information as the optimal design, but the optimization of (6) is much easier than the optimization of (4).

Moreover, Table 5 provides the maximin efficient equidistant and non-equidistant designs, their maximin efficiencies and the relative efficiency of the maximin efficient equidistant designs with respect to the maximin efficient non-equidistant designs for $I = 2$ and some given lower and upper bounds. Here it becomes apparent that the advantage of a maximin efficient non-equidistant design is higher when the interval $[L, U]$ gets larger.

**Table 4** The efficiency of the optimal equidistant design with respect to the optimal non-equidistant design

| $I$ | 1 | 5 | 10 | 15 | 20 | 25 | 40 | 50 |
|---|---|---|---|---|---|---|---|---|
| $g(I)$ | 1.0000 | 0.9835 | 0.9896 | 0.9930 | 0.9949 | 0.9961 | 0.9979 | 0.9984 |

**Table 5** Maximin efficient equidistant and non-equidistant designs for $I = 2$ with their maximin efficiencies and the relative efficiency

| | | Non-equidistant | | | Equidistant | | Relative efficiency |
|---|---|---|---|---|---|---|---|
| $L$ | $U$ | $\tau_1^*([L,U])$ | $\tau_2^*([L,U])$ | Maximin efficiency | $\Delta_{L,U}^*$ | Maximin efficiency | |
| 2 | 5 | 0.2854 | 0.8416 | 0.7658 | 0.3706 | 0.7384 | 0.9643 |
| 1 | 5 | 0.3169 | 1.2800 | 0.6864 | 0.4919 | 0.6137 | 0.8940 |
| 0.5 | 5 | 0.3189 | 2.3693 | 0.6484 | 0.6298 | 0.4635 | 0.7148 |
| 0.1 | 5 | 0.5947 | 6.7282 | 0.4764 | 0.9851 | 0.1787 | 0.3751 |
| 0.05 | 5 | 0.6816 | 10.3798 | 0.3977 | 1.1439 | 0.1081 | 0.2718 |

## 6 Discussion

We characterized locally optimal and maximin efficient equidistant and non-equidistant inspection times. In particular, we showed that locally optimal equidistant inspection times are almost as efficient as locally optimal non-equidistant inspection times. However, this does not hold for maximin efficient designs when the parameter space is large. This is due to a much larger inspection region in the non-equidistant case (see Table 5). However, large inspection regions can cause problems in practical applications.

For example, our research was motivated by a cooperation with mechanical engineers who were interested in the lifetime of diamonds on a drilling tool. Thereby, at given inspection times, it was checked whether the diamonds on the drilling tool were broken out or not. The broken diamonds were detected by analyzing the surface of the tool with a microscope. This is time consuming so that not too many inspection times should be used. Moreover, an additional requirement was a very small inspection region $[0, \tau]$. Then, not only the inspection times but also the number $I$ of inspections must be optimized so that $\tau_I^* \leq \tau$. The analysis of the dependence of an optimal number $I$ and optimal inspection times on the time horizon $\tau$ will be treated in another paper.

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