Consistency and robustness of tests and estimators based on depth

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Abstract

In this article it is shown that data depth does not only provide consistent and robust estimators but also consistent and robust tests. Thereby, consistency of a test means that the Type I (α) error and the Type II (β) error converge to zero with growing sample size in the interior of the nullhypothesis and the alternative, respectively. Robustness is measured by the breakdown point which depends here on a so-called concentration parameter. The consistency and robustness properties are shown for cases where the parameter of maximum depth is a biased estimator and has to be corrected. This bias is a disadvantage for estimation but an advantage for testing. It causes that the corresponding simplicial depth is not a degenerated U-statistic so that tests can be derived easily. However, the straightforward tests have a very poor power although they are asymptotic α-level tests. To improve the power, a new method is presented to modify these tests so that even consistency of the modified tests is achieved. Examples of two-dimensional copulas and the Weibull distribution show the applicability of the new method.

Keywords: breakdown point, consistency, data depth, Gaussian copula, Gumbel copula, likelihood depth, parametric estimation, robustness, simplicial depth, tests, Weibull distribution

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1 Introduction

Most approaches in robustness against outliers and contamination concern estimation. The absolute minority deals with robust testing, see e.g. the books on robust statistics of Jurecková and Sen (1996), Müller (1997), Maronna et al. (2006), Huber and Ronchetti (2009), Heritier et al. (2009). Robust tests are typically asymptotic α level tests, since it is impossible to derive the finite sample distribution. Moreover, the behavior under the alternative is unknown. Even consistency is usually not proved for these robust tests, although this is an important property of a test. Thereby a test $\varphi_N : \mathcal{Z}^N \to \{0,1\}$ for $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta \setminus \Theta_0$ is called consistent, if

$$\lim_{N\to\infty} P_\theta(\varphi_N = 1) = 0 \text{ for all } \theta \in \text{int}(\Theta_0),$$
$$\lim_{N\to\infty} P_\theta(\varphi_N = 1) = 1 \text{ for all } \theta \in \text{int}(\Theta \setminus \Theta_0),$$

where $\text{int}(A)$ denotes the interior of a set A. Hence consistency of a test concerns the Type I as well as the Type II error. In this article, we prove the consistency of some robust tests based on data depth. Since consistency and robustness of a test are strongly related to consistency and robustness of corresponding estimators, we also treat both for estimators based on data depth.

The concept of data depth is an approach to generalize the median and ranks to multivariate data and more complex situations. Notions of data depth were for example developed for multivariate data by Tukey (1975), Liu (1990) and Mosler (2002). Meanwhile data depth is a wide spread method with many applications, see e.g. Lin and Chen (2006), Li and Liu (2008), Romanazzi (2009), López-Pintado and Romo (2009), López-Pintado et al. (2010), Hu et al. (2009) for some recent applications. Zuo and Serfling (2000a,b) provided some general properties of depth notions. An important step for more generalizations was the development of the concept of the nonfit. Depth notions defined via the nonfit were at first developed by Rousseeuw and Hubert (1999) for regression. Mizera (2002) extended this approach by using general quality functions for the nonfit and by distinguishing between global and tangent depth. Quality functions can be obtained in particular by likelihood functions. This leads to the concept of likelihood depth which can be applied to general situations. For example, it was applied to location and scale depth in Mizera and Müller (2004). Estimators maximizing the likelihood depth are robust alternatives to the nonrobust maximum likelihood estimators (MLE). The estimators based on likelihood depth are comparable concerning flexibility and applicability to the MLE. The latter is depending very much on the underlying distribution, small changes in the distribution can lead to extreme adulterations in the estimator. This is not the case for the estimators maximizing the likelihood depth. In particular, in simulation studies these estimators showed high robustness against contaminations with other distributions, but the robustness was not proved formally yet. See e.g. Wellmann et al. (2007) and Denecke and Müller (2011).

Any depth notion can be used to define simplicial depth as Liu (1990) did using the halfspace depth of Tukey (1975). Every simplicial depth is an U-statistic so that its asymptotic distribution is in principle known and asymptotic α-level tests can be derived, see Müller (2005). In particular, general hypotheses of the form $H_0 : \theta \in \Theta_0$ can be tested by using the maximum simplicial depth over the set Θ_0. However, many simplicial depth notions are degenerated U-statistics so that the spectral decomposition of a conditional expectation is needed to derive the asymptotic distribution which in this case is not a normal distribution. Such spectral decompositions were for example derived for polynomial regression, multiple regression, and orthogonal regression by Wellmann et al. (2009) and Wellmann and Müller (2010a,b), respectively. There are also cases where the simplicial depth is not a degenerated U-statistic, for example the simplicial depth based on likelihood depth for copulas, as Denecke and Müller (2011) showed. Nondegenerated U-statistics have the normal distribution as asymptotic distribution so that asymptotic α-level tests can be derived easily. However, the price for a nondegenerated simplicial is a bias of the estimator maximizing the underlying depth notion. This bias caused also a very bad asymptotic power of the tests for some alternatives in simulation studies (see Denecke and Müller 2011).

In this article, we provide a robustness proof and an explanation for the empirical observed bad power at some alternatives. In particular, we prove that the straightforward tests are not consistent but that they can be corrected so that they are consistent. Further, we show that the correction is different from the correction which is needed to obtain the consistency of the corresponding maximum depth estimators. But they are related as well as the robustness properties. Therefore we also treat the consistency and robustness of estimators based on maximum depth. The robustness is derived via the breakdown point. Since we also allow bounded parameter space, it is shown that the breakdown point depends on a so-called concentration parameter which counts the number of observations in special subsets of the observation space. These subsets can be empty sets, single points, linear subspaces or larger subsets.

The consistency and robustness of estimators and tests are shown for depth based on general quality functions and one-dimensional parameters. We think that this approach also can be used for multidimensional parameters, but this needs much more technical effort. In particular, it is more difficult to derive examples. Therefore we only sketch the possible extension for multidimensional parameters. The examples considered here are derived via likelihood depth. These examples concern univariate data coming from a Weibull distribution with known shape or known scale and bivariate data given by the Gumbel or Gaussian copula. The concentration parameter of these examples are based on subsets which are a empty set (scale parameter of the Weibull distribution), a single point (shape parameter of the Weibull distribution), linear subspaces (Gaussian copula) and also a larger subset (Gumbel copula).

The paper is organized as follows. Section 2 concerns the consistency and the breakdown point of estimators based on data depth. It presents the new characterization of the breakdown point via the concentration parameter. This result together with the consistency of the estimators provides the notations and foundations which are needed to derive the consistency and robustness of the tests in Section 3. In particular, the results for the examples are needed to derive later consistent and robust tests for these examples. The completely new approach for testing is presented in Section 3. It provides consistency and robustness of tests based on depth for hypotheses given by $H_0 : \theta \leq \theta_0$, $H_0 : \theta \geq \theta_0$, and $H_0 : \theta = \theta_0$. Thereby, also the inconsistency of the straightforward tests is proved. Each section ends by treating the examples of Weibull distribution and the copulas. Before the examples are given, a sketch of the possible extension for multidimensional parameters is given in each of Section 2 and 3. All proofs can be found in the Appendix.

2 Estimators based on depth

2.1 Depth and maximum depth estimators

Let $Z_1, \dots Z_N$ be i.i.d. random variables with values in $\mathcal{Z} \subset \mathbb{R}^p$ and with continuous distribution P_θ and density f_θ, $\theta \in \Theta \subset \mathbb{R}^q$. As in Mizera (2002) let $Q : \Theta \times \mathcal{Z} \to \mathbb{R}$ denote a quality function that measures for every data z_n, $n = 1, \dots, N$, how well θ fits the data. The tangential depth for θ in data $z_* = (z_1, \dots, z_N)$, see Mizera (2002), then can be defined as

$$d_T(\theta, z_*) := \frac{1}{N} \inf_{u \neq 0} \sharp\{n; u^\top \nabla_\theta Q(\theta, z_n) \leq 0\},$$

where ∇_θ denotes the gradient with respect to θ. The *maximum depth estimator (MD estimator)* at z_* is then the parameter $\tilde{\theta}_N(z_*)$ which has maximum depth, i.e.

$$\tilde{\theta}_N(z_*) \in \arg\max_{\theta\in\Theta} d_T(\theta, z_*).$$

We are going to treat the case, when θ is one-dimensional. In this situation, the tangent depth has the following simple form:

$$d_T(\theta, z_*) = \min\left(\lambda_N^+(\theta, z_*), \lambda_N^-(\theta, z_*)\right)$$

where

$$\lambda_N^+(\theta, z_*) := \tfrac{1}{N}\sharp\{n; z_n \in T_{pos}^\theta\},\ \lambda_N^-(\theta, z_*) := \tfrac{1}{N}\sharp\{n; z_n \in T_{neg}^\theta\}),$$

with

$$T_{pos}^\theta := \{z \in \mathcal{Z}; \tfrac{\partial}{\partial\theta}Q(\theta, z) \geq 0\}, \quad T_{neg}^\theta = \{z \in \mathcal{Z}; \tfrac{\partial}{\partial\theta}Q(\theta, z) \leq 0\}.$$

Thus, the population versions of tangent depth and maximum depth estimators for a probability measure P are given by

$$d_T(\theta, P) = \min\left(P(T_{pos}^\theta), P(T_{neg}^\theta)\right)$$

and

$$\tilde{\theta}(P) \in \arg\max_{\theta\in\Theta} d_T(\theta, P),$$

respectively.

2.2 Consistency and robustness of the estimators

Now assume

$$p_{\theta,\theta'} := P_\theta(T_{pos}^{\theta'}) = 1 - P_\theta(T_{neg}^{\theta'}),$$

which is normally the case for continuous distributions P_θ. Since $d_T(\theta, Z_*)$ converges to $d_T(\theta, P)$ with growing sample size N, the MD estimator can only be a consistent estimator for θ at P_θ if

$$P_\theta(T_{pos}^\theta) = 1 - P_\theta(T_{neg}^\theta) = \frac{1}{2}.$$

As soon as this is not satisfied, the MD estimator is biased. Then, under regularity conditions, the MD estimator will converge to a "shifted" version $s(\theta)$ of θ, where $s(\theta)$ is given by

$$p_{\theta,s(\theta)} = P_\theta(T_{pos}^{s(\theta)}) = \frac{1}{2}. \tag{1}$$

If $s : \Theta \to \Theta$ has an inverse $s^{-1} : \Theta \to \Theta$, i.e.

$$p_{s^{-1}(\theta),\theta} = P_{s^{-1}(\theta)}(T_{pos}^{\theta}) = \frac{1}{2},$$

for all $\theta \in \Theta$, and s^{-1} is continuous, then the MD estimator $\tilde{\theta}_N$ can be corrected to $s^{-1}(\tilde{\theta}_N)$ which is consistent under regularity conditions.

Standard arguments provide for example the following regularity conditions for the convergence of the MD estimator $\tilde{\theta}_N$ to $s(\theta)$ and the consistency of $s^{-1}(\tilde{\theta}_N)$.

Proposition 1. *Let P_{θ_0} be the underlying distribution, $\lambda_{\theta_0}^{+}(\theta) = P_{\theta_0}(T_{pos}^{\theta})$ and $\lambda_{\theta_0}^{-}(\theta) = P_{\theta_0}(T_{neg}^{\theta})$. If $\lambda_N^{\pm}(\cdot, Z_*)$ converges uniformly almost surely to $\lambda_{\theta_0}^{\pm}(\cdot)$, s^{-1} is continuous, and for all $\varepsilon > 0$ there exists $\delta > 0$, such that*

$$\min(\lambda_{\theta_0}^{+}(\theta), \lambda_{\theta_0}^{-}(\theta)) < \frac{1}{2} - \delta \text{ for all } \theta \text{ with } |s(\theta_0) - \theta| > \varepsilon, \tag{2}$$

then the MD estimator $\tilde{\theta}_N$ converges to $s(\theta_0)$ almost surely and the corrected MD estimator $s^{-1}(\tilde{\theta}_N)$ converges to θ_0.

To show uniform convergence of $\lambda_N^{\pm}$ we can use a generalization of the Glivenko-Cantelli-Lemma. Generalizations are for example given by the Vapnik-C̃ervonenkis classes, see e.g. van der Vaart and Wellner (1996).

An example for a Vapnik-C̃ervonenkis class is given by $\mathscr{V} = \{T_{pos}^{\theta}; \theta \in \Theta\}$, if

$$T_{pos}^{\theta} \subsetneq T_{pos}^{\theta'} \text{ and } T_{neg}^{\theta'} \subsetneq T_{neg}^{\theta} \text{ for all } \theta < \theta', \tag{3}$$

or

$$T_{pos}^{\theta} \subsetneq T_{pos}^{\theta'} \text{ and } T_{neg}^{\theta'} \subsetneq T_{neg}^{\theta} \text{ for all } \theta > \theta'. \tag{4}$$

This class shatters no set of two points so that the Vapnik-C̃ervonenkis index is two, where the Vapnik-C̃ervonenkis index of a class is the smallest n for which no set of size n is shattered by the class, see van der Vaart and Wellner (1996). Hence the convergence of $\lambda_N^{\pm}(\cdot, Z_*)$ is uniform. Further, $T_{pos}^{\theta} \subsetneq T_{pos}^{\theta'}$ for $\theta > \theta'$ implies that $\lambda_{\theta_0}^{+}(\cdot)$ is strictly decreasing and $\lambda_{\theta_0}^{-}(\cdot)$ is strictly increasing. Hence, if $\lambda_{\theta_0}^{+}(s(\theta_0)) = \frac{1}{2} = \lambda_{\theta_0}^{-}(s(\theta_0))$, then the second condition of Proposition 1 is true as $\lambda^-_{\theta_0}(\theta) < \frac{1}{2}$ for $\theta < s(\theta_0)$ and $\lambda^+_{\theta_0}(\theta) < \frac{1}{2}$ for $\theta > s(\theta_0)$. The same holds for $T^\theta_{pos} \subsetneq T^{\theta'}_{pos}$ for $\theta < \theta'$.

The conditions (3) and (4) imply also a high breakdown point of the MD estimator and the corrected MD estimator if the observations are not too much concentrated at extreme sets of the observation space $\mathcal{Z}$. Thereby the breakdown point is the smallest proportion of the observations which must be changed so that the estimator $\hat{\theta}$ is converging to a bound of the parameter space, i.e. it is defined according to Hampel et al. (1986), p. 97, by

$$\epsilon^*(\hat{\theta}, z_*) := \frac{1}{N} \min \Big\{M; \\ \text{there exists no compact set } \Theta_0 \subset \text{int}(\Theta) \text{ with } \{\hat{\theta}(\overline{z}_*);\ \overline{z}_* \in \mathcal{Y}_M(z_*)\} \subset \Theta_0 \Big\},$$

where

$$\mathcal{Y}_M(z_*) := \left\{\overline{z}_* \in \mathcal{Z}^N;\ \sharp\{n;\ z_n \neq \overline{z}_n\} \leq M\right\}$$

is the set of contaminated samples corrupted by at most M observations. Thereby $\text{int}(A)$ denotes the interior of a set A. If the parameter space is unbounded, then convergence to ∞ or $-\infty$ means also convergence to a bound of the parameter space. The concentration at extreme sets of the observation space is measured by the following concentration parameter. Thereby define $\limsup_{\theta\downarrow\inf\Theta} := \lim_{m\to\infty} \sup_{\theta\in[\inf\Theta,\inf\Theta+1/m]\cap\Theta}$ for $\inf\Theta > -\infty$ and $\limsup_{\theta\uparrow\sup\Theta} := \lim_{m\to\infty} \sup_{\theta\in[\sup\Theta-1/m,\sup\Theta]\cap\Theta}$ for $\sup\Theta < \infty$ to include open as well as closed bounded parameter spaces Θ. Then $\limsup_{\theta\downarrow\inf\Theta} \sharp\{n;\ z_n \in T^\theta_{pos}\}$, $\limsup_{\theta\uparrow\sup\Theta} \sharp\{n;\ z_n \in T^\theta_{pos}\}$, $\limsup_{\theta\downarrow\inf\Theta} \sharp\{n;\ z_n \in T^\theta_{neg}\}$ and $\limsup_{\theta\uparrow\sup\Theta} \sharp\{n;\ z_n \in T^\theta_{neg}\}$ are always elements of $\{0, 1, \ldots, N\}$.

Definition 1. *The concentration number* $\mathscr{C}$ *of the sample* $z_1, \ldots, z_N$ *is defined as*

$$\mathscr{C} := \min\left\{\max\left\{\limsup_{\theta\downarrow\inf\Theta} \sharp\{n;\ z_n \in T^\theta_{pos}\}, \limsup_{\theta\uparrow\sup\Theta} \sharp\{n;\ z_n \in T^\theta_{neg}\}\right\}, \\ \max\left\{\limsup_{\theta\downarrow\inf\Theta} \sharp\{n;\ z_n \in T^\theta_{neg}\}, \limsup_{\theta\uparrow\sup\Theta} \sharp\{n;\ z_n \in T^\theta_{pos}\}\right\}\right\}.$$

The following lemma is obvious.

Lemma 1. *If (3) or (4) holds, then*

$$\mathscr{C} = \max\left\{\sharp\{n;\ z_n \in \bigcap_{\theta\in\Theta} T^\theta_{pos}\}, \sharp\{n;\ z_n \in \bigcap_{\theta\in\Theta} T^\theta_{neg}\}\right\}.$$

The examples below show that $\bigcap_{\theta\in\Theta} T^\theta_{pos}$ and $\bigcap_{\theta\in\Theta} T^\theta_{neg}$ often are affine subspaces of the observation space $\mathcal{Z} \subset \mathbb{R}^p$, so that the concentration parameter $\mathscr{C}$ measures the concentration of the observations at these subspaces. If $\bigcap_{\theta\in\Theta} T^\theta_{pos}$ and $\bigcap_{\theta\in\Theta} T^\theta_{neg}$ are affine subspaces and the considered distributions are continuous on $\mathcal{Z}$, then these subspaces contain at most one observation so that $\mathscr{C} \le 1$. In these cases, the breakdown point is very high. This follows from the following general theorem.

Theorem 1. *If $\mathscr{C}$ is the concentration number of $z_1, \ldots, z_N$, then the breakdown point of the MD estimator $\tilde{\theta}_N$ at $z_1, \ldots, z_N$ is at least $\frac{1}{N}\left\lfloor \frac{N+1}{2} - \mathscr{C} \right\rfloor$.*

As soon as s^{-1} is continuous, the breakdown points of the MD estimator $\tilde{\theta}_N$ and the corrected MD estimator $s^{-1}(\tilde{\theta}_N)$ coincide. Hence in these cases, Theorem 1 also provides the breakdown point of the corrected MD estimator.

2.3 Multidimensional parameters

If the parameter space Θ is q-dimensional with $q > 1$, then T^θ_{pos} and T^θ_{neg} must be replaced by $T^\theta_u = \{z \in \mathcal{Z}; u^\top \nabla_\theta Q(\theta, z_n) \le 0\}$ with $0 \ne u \in \mathbb{R}^q$. Then Proposition 1 holds analogously. Also an extension of Theorem 1 is possible. But the main difficulty is to determine the shift functions $s, s^{-1} : \Theta \to \Theta$ for examples since $\Theta \subset \mathbb{R}^q$.

2.4 Examples

To provide some examples, we now consider the so-called likelihood depth, see e.g. Mizera and Müller (2004). Here the quality function is given by the log-likelihood function, i.e. $Q(\theta, z_n) = \ln L(\theta, z_n)$. The conditions (3) and (4) are often but not always satisfied. At first we give three examples where they are satisfied and then one example where they are not satisfied.

Example 1 (Weibull distribution with known shape parameter)**.** Suppose $Z_n \sim \mathrm{Wei}(a, b)$, $n = 1, \ldots, N$, where $\mathrm{Wei}(a, b)$ denotes the Weibull distribution with shape parameter $a > 0$ and scale parameter $b > 0$. The density of Z_n for $z \ge 0$ is given by

$$f_{a,b}(z) = \frac{a}{b}\left(\frac{z}{b}\right)^{a-1} \exp\left(-\left(\frac{z}{b}\right)^a\right).$$

Suppose at first that the shape parameter a is known as a_0 and the scale b is unknown. Since the derivative of the log-likelihood function,

$$\tfrac{\partial}{\partial b} \ln L(a_0, b, z) = -\frac{a_0}{b} + \frac{a_0}{b}\left(\frac{z}{b}\right)^{a_0},$$

is positive or zero, iff $z \geq b$, we have $T^b_{pos} = [b, \infty)$, $T^b_{neg} = (0, b]$, and the maximum likelihood depth (MLD) estimator is the median $med(z_*)$ of the data. But the median of the Weibull distribution Wei(a_0, b_0) is $s(b_0) = b_0 (\ln 2)^{\frac{1}{a_0}} < b_0$. Therefore, the estimator based on likelihood depth for the scale parameter of the Weibull distribution is a biased estimator. However, since $s^{-1}(b) = b \frac{1}{(\ln 2)^{\frac{1}{a_0}}}$ is continuous, we obtain the well known result that $\hat{b}(z_*) = s^{-1}(med) = \frac{1}{(\ln 2)^{\frac{1}{a_0}}} med(z_*)$ is a consistent estimator of the scale. Moreover, the concentration parameter $\mathscr{C}$ is 0 since $\bigcap_{b \in (0,\infty)} T^b_{neg} = \emptyset = \bigcap_{b \in (0,\infty)} T^b_{pos}$. Hence, according to Lemma 1 and Theorem 1, the breakdown point of the corrected MLD estimator $\hat{b}(z_*)$ is at least $\frac{1}{N} \left\lfloor \frac{N+1}{2} \right\rfloor$. This is the well known breakdown point of the median.

Example 2 (Weibull distribution with known scale parameter)**.** Now assume the scale $b_0 > 0$ to be known and the shape a to be unknown. The partial derivative of the log-likelihood function with respect to a is

$$\frac{\partial}{\partial a} \ln L(a, b_0, z) = \frac{1}{a} + \ln \frac{z}{b_0} - \ln \left(\frac{z}{b_0} \right) \left(\frac{z}{b_0} \right)^a .$$

This is positive, iff $c_1^{\frac{1}{a}} b_0 \leq z \leq c_2^{\frac{1}{a}} b_0$, where $c_1 \approx 0.259$, $c_2 \approx 2.240$ being the solutions of $\ln c = \frac{1}{c-1}$. Hence $T^a_{pos} = [c_1^{\frac{1}{a}} b_0, c_2^{\frac{1}{a}} b_0]$, $T^a_{neg} = (0, c_1^{\frac{1}{a}} b_0] \cup [c_2^{\frac{1}{a}} b_0, \infty)$, and the likelihood depth for the shape parameter is

$$d^b_T(a, z_*) = \frac{1}{N} \min \left(\sharp \left\{ n; c_1^{\frac{1}{a}} b_0 \leq z_n \leq c_2^{\frac{1}{a}} b_0 \right\}, \sharp \left\{ n; z_n \leq c_1^{\frac{1}{a}} b_0 \text{ or } z_n \geq c_2^{\frac{1}{a}} b_0 \right\} \right).$$

Because of $T^a_{pos} \subset T^{a'}_{pos}$, $T^a_{neg} \supset T^{a'}_{neg}$ for $a > a'$, and $\bigcap_{a \in (0,\infty)} T^a_{pos} = \{b_0\}$, $\bigcap_{a \in (0,\infty)} T^a_{neg} = \emptyset$, the concentration parameter $\mathscr{C}$ is given by $\sharp\{n;\ z_n = b_0\}$. Hence, the breakdown point of the MLD estimator $\tilde{a}_N$ is at least $\frac{1}{N} \left\lfloor \frac{N+1}{2} - 1 \right\rfloor$ if the observations are pairwise different. Since

$$p_{a_0,a} = P_{a_0,b_0}(T^a_{pos}) = \exp\left(-c_1^{\frac{a_0}{a}}\right) - \exp\left(-c_2^{\frac{a_0}{a}}\right)$$

we have

$$p_{a_0,a_0} = \exp(-c_1) - \exp(-c_2) \approx 0.665 > 0.5.$$

Thus, if a_0 is the parameter of the underlying distribution, the parameter with asymptotic maximum depth is

$$s(a_0) = \frac{1}{\kappa} \cdot a_0 > a_0,$$

where κ is the unique solution of $\exp\left(-c_1^\kappa\right) - \exp\left(-c_2^\kappa\right) = \frac{1}{2}$. It is $\kappa \approx 0.691$. Hence again $s^{-1}(a) = \kappa a$ is linear and thus continuous, so that $\hat{a}(z_*) = \kappa \arg\max_{a>0} d_T(a, z_*)$ is a robust and consistent estimator for a.

Remark 1. Denecke (2010) additionally showed that the results of Examples 1 and 2 can be transferred also to censored data. Moreover, if both parameters of the Weibull distribution are unknown, then one can determine at first the MLD estimator $\tilde{b}_N$ for scale, which is independent of the shape parameter. Then using this estimator instead of b_0 in determining the MLD estimator $\tilde{a}_N$ for the shape leads to an estimator $\tilde{a}_N$ with very similar behavior as that for known scale. In particular the bias is again linear. The shift parameter is then $\kappa \approx 0.757$, the unique solution of $2^{-c_1^\kappa} - 2^{-c_2^\kappa} = \frac{1}{2}$, so that $\hat{a}_N(z_*) = \frac{1}{0.757}\tilde{a}_N(z_*)$ is a consistent estimator of a. Then $\hat{b}(z_*) = \frac{1}{(\ln 2)^{\frac{1}{\hat{a}_N(z_*)}}} med(z_*)$ is also a consistent estimator of b. Both estimators have the high breakdown point given by Theorem 1.

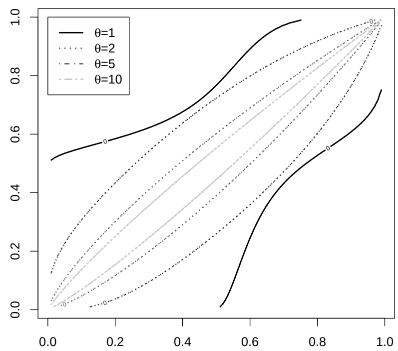

Figure 1: Zeros of $\frac{\partial}{\partial\theta}L(\theta,\cdot)$ for the Gumbel copula with different θ.

Example 3 (Gumbel copula)**.** The Gumbel copula is an Archimedian copula defined by

$$C_\theta : [0,1]\times[0,1] \to [0,1], \qquad (u,v) \mapsto C_\theta(u,v) := \exp\left(-((-\ln u)^\theta + (-\ln v)^\theta)^{\frac{1}{\theta}}\right)$$

for $\theta \geq 1$. Using that the density of the Gumbel copula is

$$f_\theta(z) = \frac{(-\ln v)^{\theta-1}(-\ln u)^{\theta-1}}{uv} e^{-((-\ln u)^\theta + (-\ln v)^\theta)^{\frac{1}{\theta}}} \left(\theta - 1 + ((-\ln u)^\theta + (-\ln v)^\theta)^{\frac{1}{\theta}}\right)((-\ln u)^\theta + (-\ln v)^\theta)^{\frac{1}{\theta}-2}$$

for $z = (u,v) \in [0,1]\times[0,1]$, $\theta \geq 1$, the likelihood depth of a parameter θ in data with dependence structure given by the Gumbel copula and margins being uniform on $[0,1]$ can be calculated. In Denecke and Müller (2011), the asymptotic bias of the MLD estimator was calculated numerically as $s(\theta) \approx 1.408\,\theta - 0.021$ so that the corrected estimator is

$s^{-1}(\tilde{\theta}_N) \approx 0.71\,\tilde{\theta}_N + 0.015$. The robustness was shown only by a simulation study. But no proofs of consistency and robustness were given. Now proofs can be given with the following arguments.

To determine T^{θ}_{pos} for the Gumbel copula, the zeros of $\frac{\partial}{\partial\theta}\ln L(\theta,\cdot)$ are needed (they are the boundaries of T^{θ}_{pos}). We could not find explicit algebraic expressions for them, so we had a look at the contour-plot of $\frac{\partial}{\partial\theta}\ln L(\theta,u,v)$ for fixed θ and variable $u,v\in[0,1]$. The zeros for different θ are displayed in Figure 1. The area where $\frac{\partial}{\partial\theta}\ln L(\theta,\cdot)$ is positive (T^{θ}_{pos}) lies between the zeros. Thus, the graphic leads to the proposal that for $\theta' < \theta$ it holds $T^{\theta}_{pos} \subset T^{\theta'}_{pos}$. Consequently the maximum likelihood depth estimator for the parameter θ of the Gumbel copula should be a strongly consistent estimator for $s(\theta)$. Additionally, it holds $\bigcap_{\theta\in[1,\infty)} T^{\theta}_{pos} = \{(x,y)\in[0,1]^2;\ x=y\}$ and $\bigcap_{\theta\in[1,\infty)} T^{\theta}_{neg} = T^{1}_{neg}$. Especially $\bigcap_{\theta\in[1,\infty)} T^{\theta}_{neg}$ here is large so that also the concentration parameter $\mathscr{C}$ can be large, in particular for distributions close to independence.

Example 4 (Gaussian copula)**.** As a second copula we consider the Gaussian copula with standard normal margins. The *Gaussian copula* is defined as

$$C_\rho(u,v) := \int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}}\exp\left(\frac{-(s^2-2\rho st+t^2)}{2(1-\rho^2)}\right) ds\,dt,$$

$u,v\in(0,1), -1<\rho<1$, where Φ denotes the one-dimensional standard normal distribution function. In Denecke and Müller (2011), it was shown that here the asymptotic bias s is not linear as in the case of the Gumbel copula. Hence s^{-1} is also not linear and was numerically calculated as $s^{-1}(\tilde{\rho}_N) \approx -1.22217\,\tilde{\rho}_N^3 + 3.6434\,\tilde{\rho}_N^2 - 1.42154\,\tilde{\rho}_N + 0.000396$. Simulations in Denecke and Müller (2011) provided also the robustness of this corrected MLD estimator, but no proofs were given. Now proofs of the consistency and a high breakdown point are possible.

At first note that

$$T^{\rho}_{pos} = \{(u,v);\ v_2(u,\rho)\le v\le v_1(u,\rho)\},\ \text{for } \rho>0,$$

and

$$T^{\rho}_{pos} = \{(u,v); v\le v_1(u,\rho) \text{ or } v\ge v_2(u,\rho)\},\ \text{for } \rho<0,$$

where $v_{1,2}(u,\rho) = u\frac{1+\rho^2}{2\rho} \pm \frac{\sqrt{\rho^4u^2-2\rho^2u^2+u^2-4\rho^4+4\rho^2}}{2\rho}$ are such that $\frac{\partial}{\partial\rho}\ln f_\rho(x, v_i(u,\rho)) = 0$, $i=1,2$ with f_ρ the density of the two-dimensional normal distribution.

The conditions (3) and (4) are not satisfied. In particular, $\lambda^{+}_{\rho_0}(\cdot)$ is not decreasing but is a convex function that takes the value $\frac{1}{2}$ only once for $\rho\neq 0$, see also Denecke (2010). But condition (2) is satisfied for $\rho\neq 0$. Moreover, $\mathscr{V} = \{T^{\rho}_{pos}; \rho\in(-1,1)\}$ is also a VC-class, namely a VC-class with a Vapnik-Červonenkis index $V(\mathscr{V}) < 7$. This was shown in Denecke (2010) and it was used that for every $z = (u, v)$, there exist at most 9 intervals of type $[\rho_1, \rho_2]$, such that $z \in T^{\rho}_{pos}$ for $\rho \in [\rho_1, \rho_2]$. Hence, according to Proposition 1, the corrected MLD estimator $s^{-1}(\tilde{\rho}_N)$ is a consistent estimator of ρ.

Although the conditions (3) and (4) are not satisfied for all ρ, ρ', they are satisfied for $\rho, \rho' \geq 0.5$ and for $\rho, \rho' \leq -0.5$. As $v_1(u, \cdot)$ is strictly decreasing for $\rho \geq 0.5$ and $\rho \leq -0.5$ and $v_2(u, \cdot)$ is strictly increasing for $\rho \geq 0.5$ and $\rho \leq -0.5$, we have (3) for $\rho \geq 0.5$ and $\rho \leq -0.5$. This means that the concentration parameter $\mathscr{C}$ here also is given by Lemma 1. It holds $\bigcap_{\rho\in(-1,1)} T^{\rho}_{pos} = \{(u, v);\ u = v\}$, as $\lim_{\rho\to 1} T^{\rho}_{pos} = \{(u, v);\ u = v\}$, and $\bigcap_{\rho\in(-1,1)} T^{\rho}_{neg} = \{(u, v);\ u = -v\}$, as $\lim_{\rho\to -1} T^{\rho}_{neg} = \{(u, v);\ u = -v\}$. Since the probability of observations lying in a subspace is zero, the concentration parameter $\mathscr{C}$ should be small implying a high breakdown point according to Theorem 1, as long as we do not consider contamination with total dependent data.

3 Tests based on depth

3.1 Robust tests based on simplicial depth

For testing we use the simplicial depth, see e.g. Müller (2005). Thereby, the *simplicial depth* of θ within observations $z_* := (z_1, \ldots, z_N)^T$ is defined as

$$d_S(\theta, z_*) := \binom{N}{q+1}^{-1} \sharp\{\{n_1, \ldots, n_{q+1}\} \subset \{1, \ldots, N\}; d_T(\theta, (z_{n_1}, \ldots, z_{n_{q+1}})) > 0\},$$

where q is the dimension of θ.

If $\theta \in \mathbb{R}$, then

$$d_S(\theta, z_*) = \frac{1}{\binom{N}{2}} \sum_{1\leq n_1 < n_2 \leq N} d_T(\theta, (z_{n_1}, z_{n_2})).$$

In this case, the tangent depth of two observations z_{n_1}, z_{n_2} is non-zero if and only if $\frac{\partial}{\partial\theta}Q(\theta, z_{n_1})\frac{\partial}{\partial\theta}Q(\theta, z_{n_1}) \leq 0$. Assuming again $P_\theta(\{z;\ \frac{\partial}{\partial\theta}Q(\theta, z) = 0\}) = 0$, the simplicial depth satisfies asymptotically

$$\begin{aligned} d_S(\theta, z_*) &\cong \frac{2}{N(N-1)} \sum_{1\leq n_1 < n_2 \leq N} \left(1_{T^{\theta}_{pos}}(z_{n_1}) 1_{T^{\theta}_{neg}}(z_{n_2}) + 1_{T^{\theta}_{neg}}(z_{n_1}) 1_{T^{\theta}_{pos}}(z_{n_2})\right) &(5)\\ &\cong 2 \cdot \lambda^+_N(\theta, z_*) \cdot \lambda^-_N(\theta, z_*) &(6) \end{aligned}$$

where 1_A denotes the indicator function of A, namely $1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$.

Characterization (6) implies for large N that the simplicial depth is maximized if $\lambda^+_N(\theta, z_*) = \lambda^-_N(\theta, z_*)$. But under this condition the tangent depth

$$d_T(\theta, z_*) = \min\left(\lambda^+_N(\theta, z_*), \lambda^-_N(\theta, z_*)\right)$$

is also maximized so that maximum depth estimators based on tangent depth coincide asymptotically with those based on simplicial depth. Hence their breakdown points coincide asymptotically and are given in particular by Theorem 1. But the simplicial depth has also itself a high breakdown point. Assuming $\lambda_N^+(\theta, z_*) = 1 - \lambda_N^-(\theta, z_*)$, characterization (6) provides $d_S(\theta, z_*) \in [0, \frac{1}{2}]$ for large N. If a sample z_* provides a simplicial depth close to zero, i.e. $\lambda_N^+(\theta, z_*) \approx 0, \lambda_N^-(\theta, z_*) \approx 1$ or $\lambda_N^+(\theta, z_*) \approx 1, \lambda_N^-(\theta, z_*) \approx 0$, than approximately half of the observation must be changed to obtain $d_S(\theta, z_*) \approx \frac{1}{2}$, i.e. $\lambda_N^+(\theta, z_*) \approx \frac{1}{2}, \lambda_N^-(\theta, z_*) \approx \frac{1}{2}$. Vice versa, if the simplicial depth is close to $\frac{1}{2}$, then approximately half of the observation must be changed to obtain a simplicial depth close to 0. Hence using the simplicial depth in a test statistic will lead to a test with high breakdown point in the sense of Ylvisaker (1977), He et al. (1990), Coakley and Hettmansperger (1994), Zhang (1996), Müller (1997).

Characterization (5) means that the simplicial depth is a U-statistic with

$$\begin{aligned} & \mathrm{E}\left(1_{T^\theta_{pos}}(Z_1)1_{T^\theta_{neg}}(Z_2) + 1_{T^\theta_{neg}}(Z_1)1_{T^\theta_{pos}}(Z_2)\middle|\, Z_1 = z_1\right) \\ & \quad = \quad (1 - p_{\theta,\theta})1_{T^\theta_{pos}}(z_1) + p_{\theta,\theta}1_{T^\theta_{neg}}(z_1). \end{aligned}$$

Hence the simplicial depth is a nondegenerated U-statistics if and only if $p_{\theta,\theta} = P_\theta(T^\theta_{pos}) \neq \frac{1}{2}$. According to the theorem of Hoeffding, see e.g. Lee (1990), the asymptotic distribution of a nondegenerated U-statistics is the normal distribution. Hence a test statistic based on

$$T(\theta, z_*) := \sqrt{N}\frac{d_S(\theta, z_*) - 2p_{\theta,\theta}(1 - p_{\theta,\theta})}{2\sqrt{(1 - p_{\theta,\theta})p_{\theta,\theta}(1 - 2p_{\theta,\theta})^2}} \tag{7}$$

satisfies $T(\theta, Z_{*,N}) \xrightarrow{\mathscr{D}} X \sim \mathcal{N}(0, 1)$ for all $\theta \in \Theta$ with $p_{\theta,\theta} = P_\theta(T^\theta_{pos}) \neq \frac{1}{2}$ (see Corollary 1 in the Appendix). Hence the asymptotic bias of the MD estimator implies a simple asymptotic distribution of the test statistic (7). Since we regard here only the case $p_{\theta,\theta} = P_\theta(T^\theta_{pos}) \neq \frac{1}{2}$, asymptotic tests with level α can be based on this test statistic. In particular, as for simplicial depths which are nondegenerated U-statistics (see Müller 2005), tests defined by

$$\varphi(z_*) := 1_{\{\sup_{\theta \in \Theta_0} T(\theta, z_*) < \Phi^{-1}(\alpha)\}}(z_*)$$

are asymptotic α-level tests for $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \notin \Theta_0$, i.e. they satisfy $\lim\limits_{N\to\infty} P_\theta\left(\varphi(Z_{*,N}) = 1\right) \leq \alpha$ for all $\theta \in \Theta_0$. Such tests reject the null hypothesis if the maximum simplicial depth of parameters of the null hypothesis is too small.

This leads to various tests based on depth. Especially we have that

$$\varphi_{\theta_0}^{0,=}(z_*) := 1_{\{T(\theta_0,z_*)<\Phi^{-1}(\alpha)\}}(z_*) \tag{8}$$

is an asymptotic α-level test for $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$,

$$\varphi_{\theta_0}^{0,\geq}(z_*) := 1_{\{\sup_{\theta\geq\theta_0} T(\theta,z_*)<\Phi^{-1}(\alpha)\}}(z_*) \tag{9}$$

is an asymptotic α-level test for $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$, and

$$\varphi_{\theta_0}^{0,\leq}(z_*) := 1_{\{\sup_{\theta\leq\theta_0} T(\theta,z_*)<\Phi^{-1}(\alpha)\}}(z_*) \tag{10}$$

is an asymptotic α-level test for $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$.

Since these tests work only if the MD estimator is asymptotically biased, we study the asymptotic power and the consistency of these tests separately for the two cases $s(\theta_0) > \theta_0$ and $s(\theta_0) < \theta_0$, i.e. where the MD estimator overestimates the true value θ_0 and where MD estimator underestimates the true value θ_0, respectively. We show that these tests are not consistent but can be corrected so that they are consistent.

3.2 Consistency of the tests for the case $s(\theta_0) > \theta_0$

If the MD estimator overestimates the true parameter θ, i.e. $s(\theta_0) > \theta_0$, then the test $\varphi_{\theta_0}^{0,\geq}(z_*) := 1_{\{\sup_{\theta\geq\theta_0} T(\theta,z_*)<\Phi^{-1}(\alpha)\}}(z_*)$ for testing $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$ has a bad power. The reason is that there exists $\theta' < \theta_0$ with $s(\theta') > \theta_0$ so that distributions with θ from the alternative produce high values of the simplicial depth and thus of the test statistic (7). Thereby recall that the tangent depth and the simplicial depth attain their maximum value at the same parameter for large N. To improve the power, the rejection set $\{\sup_{\theta\geq\theta_0} T(\theta, z_*) < \Phi^{-1}(\alpha)\}$ must be extended. This can be done by taking the supremum over a smaller set, i.e. $\theta \geq c_\alpha^1(\theta_0) > \theta_0$ instead of $\theta \geq \theta_0$. To obtain the best power, $c_\alpha^1(\theta_0)$ should be chosen as the maximum value where the resulting test is still an asymptotic α-level test. Hence $c_\alpha^1(\theta_0)$ should be chosen as

$$c_\alpha^1(\theta_0) := \max\{\theta; \lim_{N\to\infty} P_{\theta_0}\left(T(\theta, Z_{*,N}) < \Phi^{-1}(\alpha)\right) \leq \alpha\}.$$

Lemma 4 in the Appendix provides under the assumptions

$$p_{(\cdot),\theta} = P_{(\cdot)}(T_{pos}^\theta) \text{ is strictly increasing from 0 to 1,} \tag{11}$$

$$p_{\theta,(\cdot)} \text{ is strictly decreasing,} \tag{12}$$

$$\frac{1}{2} < p_{\theta,\theta} \leq \frac{1}{2} + \frac{1}{\sqrt{8}}, \text{ and} \tag{13}$$

$$\alpha < 0.5, \tag{14}$$

for $\theta = \theta_0$ that $c_\alpha^1(\theta_0)$ is the value $\tilde{\theta}$, such that

$$1 - p_{\tilde{\theta},\tilde{\theta}} = p_{\theta_0,\tilde{\theta}}.$$

Note that condition (4) implies condition (12).

If $\frac{1}{2} < p_{\theta,\theta}$ holds for all θ, then $p_{\theta_0,c_\alpha^1(\theta_0)} < \frac{1}{2}$. This means in particular that $c_\alpha^1(\theta_0) > s(\theta_0) > \theta_0$ since $s(\theta_0)$ is that value with $p_{\theta_0,s(\theta_0)} = \frac{1}{2}$ according to (1).

The corrected test is now given by

$$\varphi_{\theta_0}^{\geq}(z_*) := 1_{\{\sup_{\theta \geq c_\alpha^1(\theta_0)} T(\theta,z_*) < \Phi^{-1}(\alpha)\}}(z_*).$$

To compare the asymptotic power of this test with the power of the original test $\varphi_{\theta_0}^{0,\geq}(z_*)$, we need also the inverse function of $c_\alpha^1(\theta_0)$, i.e. a function $\check{c}_\alpha^1(\theta_0)$ with $c_\alpha^1(\check{c}_\alpha^1(\theta_0)) = \theta_0$ and $\check{c}_\alpha^1(c_\alpha^1(\theta_0)) = \theta_0$. This function is given by

$$\check{c}_\alpha^1(\theta_0) := \min\{\theta; \lim_{N\to\infty} P_\theta\left(T(\theta_0, Z_{*,N}) < \Phi^{-1}(\alpha)\right) \leq \alpha\},$$

or (see Lemma 4 in the Appendix) by that value $\tilde{\theta}$, such that

$$1 - p_{\theta_0,\theta_0} = p_{\tilde{\theta},\theta_0},$$

so that $\check{c}_\alpha^1(\theta_0) < s^{-1}(\theta_0) < \theta_0$.

Additionally assume

$$c_\alpha^1 \text{ is an increasing function,} \tag{15}$$

$$p_{\theta,\theta} \text{ is a continuous function of } \theta. \tag{16}$$

Theorem 2. *Let the conditions (11)-(16) be valid for all $\theta \geq \theta_0$ and $p_{\theta,\theta} \neq \frac{1}{2}$ for all $\theta \in \Theta$.*

(a) It holds

$$\lim_{N\to\infty} P_\theta\left(\varphi_{\theta_0}^{0,\geq}(Z_{*,N}) = 1\right) = \lim_{N\to\infty} P_\theta\left(\sup_{\tilde{\theta}\geq\theta_0} T(\tilde{\theta}, Z_{*,N}) < \Phi^{-1}(\alpha)\right)$$
$$\begin{cases} = 1, & \theta < \check{c}_\alpha^1(\theta_0) \\ \leq \alpha, & \theta = \check{c}_\alpha^1(\theta_0) \\ = 0, & \theta > \check{c}_\alpha^1(\theta_0) \end{cases}.$$

(b) $\varphi_{\theta_0}^{\geq}$ *is a consistent test with asymptotic level* α *for* $H_0 : \theta \geq \theta_0$*, i.e.*

$$\lim_{N\to\infty} P_\theta \left(\varphi_{\theta_0}^{\geq}(Z_{*,N}) = 1\right) = \lim_{N\to\infty} P_\theta \left(\sup_{\tilde{\theta}\geq c_\alpha^1(\theta_0)} T(\tilde{\theta}, Z_{*,N}) < \Phi^{-1}(\alpha) \right)$$

$$\begin{cases} = 1, & \theta < \theta_0 \\ \leq \alpha, & \theta = \theta_0 \\ = 0, & \theta > \theta_0 \end{cases}.$$

However, $\varphi_{\theta_0}^{0,\leq}$ needs not to be corrected.

Theorem 3. *If the conditions (11)-(16) are valid for all* $\theta \leq \theta_0$ *and* $p_{\theta,\theta} \neq \frac{1}{2}$ *for all* $\theta \in \Theta$*, then* $\varphi_{\theta_0}^{0,\leq}$ *is consistent test with asymptotic level* α *for* $H_0 : \theta \leq \theta_0$*, i.e.*

$$\lim_{N\to\infty} P_\theta \left(\varphi_{\theta_0}^{0,\leq}(Z_{*,N}) = 1\right) \begin{cases} = 0, & \theta < \theta_0 \\ \leq \alpha, & \theta = \theta_0 \\ = 1, & \theta > \theta_0 \end{cases}.$$

Theorems 2 and 3 imply that the test $\varphi_{\theta_0}^{0,=}$ for $H_0 : \theta = \theta_0$ has a bad power for $\theta < \theta_0$ and a good power for $\theta > \theta_0$. Its correction is given by

$$\varphi_{\theta_0}^{=}(z_*) := \max\{1_{\{T(\theta_0,z_*)<\Phi^{-1}\left(\frac{\alpha}{2}\right)\}}(z_*), 1_{\{T(c_{\frac{\alpha}{2}}^1(\theta_0),z_*)<\Phi^{-1}\left(\frac{\alpha}{2}\right)\}}(z_*)\}.$$

Since the null hypothesis consists only of one point, we do not need here condition (16) for the asymptotic behavior.

Theorem 4. *Let the conditions (11)-(15) be valid for* $\theta = \theta_0$ *and* $p_{\theta,\theta} \neq \frac{1}{2}$ *for all* $\theta \in \Theta$.

(a) *It holds*

$$\lim_{N\to\infty} P_\theta \left(\varphi_{\theta_0}^{0,=}(Z_{*,N}) = 1\right) = \lim_{N\to\infty} P_\theta \left(T(\theta_0, Z_{*,N}) < \Phi^{-1}(\alpha)\right)$$

$$= \begin{cases} = 1, & \theta < \check{c}_\alpha^1(\theta_0) \\ = \alpha, & \theta = \check{c}_\alpha^1(\theta_0) \\ = 0, & \check{c}_\alpha^1(\theta_0) < \theta < \theta_0 \\ = \alpha, & \theta = \theta_0 \\ = 1, & \theta > \theta_0 \end{cases}.$$

(b) $\varphi_{\theta_0}^{=}$ *is a consistent test with asymptotic level* α *for* $H_0 : \theta = \theta_0$*, i.e.*

$$\lim_{N\to\infty} P_\theta \left(\max\{1_{\{T(\theta_0,Z_*)<\Phi^{-1}\left(\frac{\alpha}{2}\right)\}}(Z_*), 1_{\{T(c_{\frac{\alpha}{2}}^1(\theta_0),Z_*)<\Phi^{-1}\left(\frac{\alpha}{2}\right)\}}(Z_*)\} \right)$$

$$= \lim_{N\to\infty} P_\theta \left(\varphi_{\theta_0}^{=}(Z_{*,N}) = 1\right) \begin{cases} = 1, & \theta \neq \theta_0 \\ \leq \alpha, & \theta = \theta_0 \end{cases}.$$

3.3 Consistency of the tests for the case $s(\theta_0) < \theta_0$

Analogous results hold for $s(\theta_0) < \theta_0$. Here $\varphi_{\theta_0}^{0,\geq}$ is already a consistent test and needs not to be corrected. However, $\varphi_{\theta_0}^{0,\leq}$ and $\varphi_{\theta_0}^{0,=}$ must be corrected to

$$\varphi_{\theta_0}^{\leq}(z_*) := 1_{\{\sup_{\theta \leq c_\alpha^2(\theta_0)} T(\theta, z_*) < \Phi^{-1}(\alpha)\}}(z_*)$$

and

$$\varphi_{\theta_0}^{=}(z_*) := \max\{1_{\{T(\theta_0, z_*) < \Phi^{-1}\left(\frac{\alpha}{2}\right)\}}(z_*), 1_{\{T(c^2_{\frac{\alpha}{2}}(\theta_0), z_*) < \Phi^{-1}\left(\frac{\alpha}{2}\right)\}}(z_*)\}.$$

Here $c_\alpha^2(\theta_0)$ is defined by

$$c_\alpha^2(\theta_0) := \min\{\theta; \lim_{N\to\infty} P_{\theta_0}\left(T(\theta, Z_{*,N}) < \Phi^{-1}(\alpha)\right) \leq \alpha\}$$

and satisfies $c_\alpha^2(\theta_0) < s(\theta_0) < \theta_0$ under conditions analogously to (11) -(14). For more details see Denecke (2010).

3.4 Multidimensional parameters

If θ is q-dimensional with $q > 1$, tests for hypotheses of the form $H_0 : h(\theta) \leq h(\theta_0)$ and $H_0 : h(\theta) = h(\theta_0)$ with $h : \Theta \to \mathbb{R}$ can be corrected as in the one-dimensional case.

3.5 Examples

Again we use the likelihood depth to provide some examples. At first we consider the Weibull distribution, followed by the Gumbel copula. The Gaussian copula is not treated here, as the conditions (11)-(16) are not fulfilled for all parameters.

Example 5 (Weibull distribution with known scale)**.** To derive consistent tests for hypotheses on the shape parameter a_0 of the Weibull distribution, with known scale parameter, recall from Example 2 that

$$p_{a_0,a} = \exp(-c_1^{\frac{a_0}{a}}) - \exp(-c_2^{\frac{a_0}{a}}).$$

The conditions (11), (12), and (14) are obviously satisfied. The condition (13) is satisfied since

$$p_{a_0,a_0} = \exp(-c_1) - \exp(-c_2) \approx 0.665 \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{\sqrt{8}}\right].$$

Thus, according to Lemma 4, we have to solve $1 - \exp(-c_1) + \exp(-c_2) = \exp(-c_1^{\frac{a_0}{a}}) - \exp(-c_2^{\frac{a_0}{a}})$ for a to determine $c_\alpha^1(a_0)$. This leads to

$$c_\alpha^1(a_0) = k_0 \cdot a_0,$$

where k_0 is the unique solution of

$$1 - \exp(-c_1) + \exp(-c_2) = \exp(-c_1^{\frac{1}{k_0}}) - \exp(-c_2^{\frac{1}{k_0}}),$$

i.e. $k_0 \approx 2.275$. Especially $c_\alpha^1(a_0)$ exists and it is $c_\alpha^1(a_0) > a_0$ for all $a_0 > 0$. Hence also the conditions (15) and (16) are satisfied so that

$$\begin{aligned}
\varphi_{a_0}^{0,\leq}(\cdot) &= 1_{\{\sup_{a \leq a_0} T(a,\cdot) < \Phi^{-1}(\alpha)\}}(\cdot), \\
\varphi_{a_0}^{\geq}(\cdot) &= 1_{\{\sup_{a \geq c_\alpha^1(a_0)} T(a,\cdot) < \Phi^{-1}(\alpha)\}}(\cdot), \\
\varphi_{a_0}^{=}(\cdot) &= \max\left(1_{\{T(a_0,\cdot) < \Phi^{-1}(\frac{\alpha}{2})\}}(\cdot), 1_{\{T(c_{\frac{\alpha}{2}}^1(a_0),\cdot) < \Phi^{-1}(\frac{\alpha}{2})\}}(\cdot)\right)
\end{aligned}$$

are consistent tests with asymptotic level α for $H_0 : a \leq a_0$, $H_0 : a \geq a_0$, $H_0 : a = a_0$, respectively. Figure 2 provides the simulated power of the tests for $H_0 : a \leq 1$, for $N = 100$ data and 1000 repetitions each, in comparison to the test based on the classical maximum likelihood estimator, see Rinne (2009), in uncontaminated data. It shows that the power of the tests based on likelihood depth is similar to that based on the maximum likelihood estimator. The power function for $H_0 : a \geq a_0$ behaves the same way. As the pictures for the tests for the shape and scale are very alike, we just show one each for uncontaminated data. On the right in Figure 2 we see the behavior of the power function of the new test for $H_0 : a = a_0$ based on likelihood depth compared to the behavior of the one based on the MLE and the one based on the method of medians, see He and Fung (1999), in contaminated data. We contaminated each sample by simulating 10% of the data coming from a Weibull distribution with shape $a_1 = 0.5$ and scale $b_1 = 5$. The power function of the test based on the maximum likelihood estimator is highly influenced by the contaminated data, while the new test based on likelihood depth and the test based on the method of medians are robust.

Example 6 (Weibull distribution with known shape)**.** Now we assume the shape parameter a_0 of the Weibull distribution to be known and derive consistent tests for hypotheses about the scale parameter. In Example 1 T_{pos}^b was determined as $[b, \infty)$, thus $p_{b_0,b} = P_{b_0}(T_{pos}^b) = \exp\left(-\left(\frac{b}{b_0}\right)^{a_0}\right)$. As $s(b_0) = b_0(\ln 2)^{\frac{1}{a_0}} < b_0$, the quantity $c_\alpha^2(b_0)$ has to be determined. Therefor an analog to Lemma 4, see Denecke (2010), can be used and we get

$$c_\alpha^2(b_0) = b_0(-\ln(1 - \exp(-1)))^{\frac{1}{a_0}} < b_0,$$

by solving $1 - p_{c_\alpha^2(b_0), c_\alpha^2(b_0)} = p_{b_0, c_\alpha^2(b_0)}$. Further, the conditions for consistency are fulfilled, as for all b_0 $p_{b_0,(\cdot)}$ is strictly decreasing, $p_{(\cdot),b_0}$ is strictly increasing, it holds $\frac{1}{2} < 1 - p_{b_0,b_0} \approx 0.623 < \frac{1}{2} + \frac{1}{\sqrt{8}}$, and $c_\alpha^2(\cdot)$ is strictly increasing. Thus,

$$\varphi_{b_0}^{0,\geq}(\cdot) = 1_{\{\sup_{b \geq b_0} T(b,\cdot) < \Phi^{-1}(\alpha)\}}(\cdot),$$

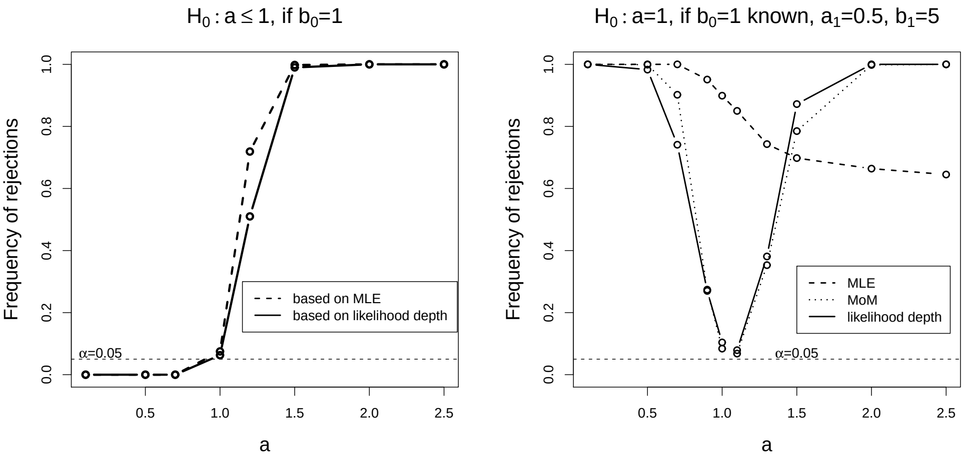

Figure 2: Simulated power of the tests for $H_0 : a \leq a_0$ in uncontaminated data and $H_0 : a = a_0$ in contaminated data, b_0 known.

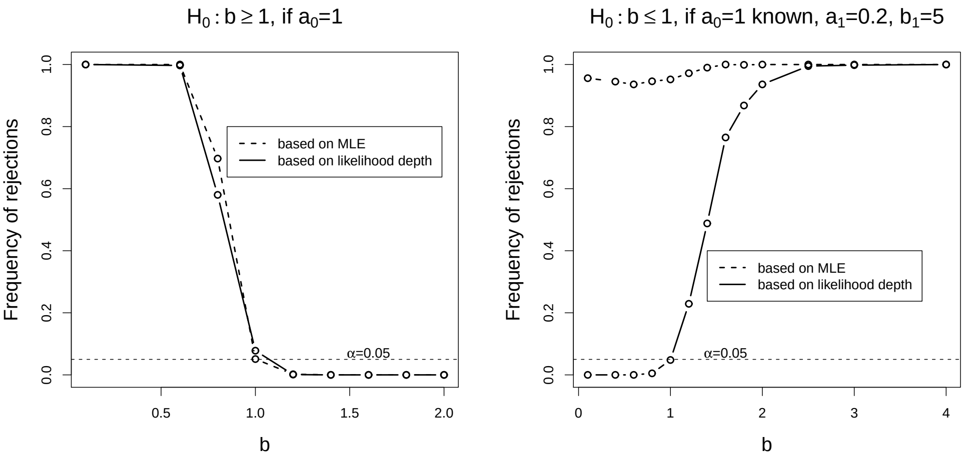

Figure 3: Simulated power of the tests for $H_0 : b \leq b_0$ with a_0 supposed to be known and $H_0 : b \geq b_0$.

$$\varphi_{b_0}^{\leq}(\cdot) = 1_{\{\sup_{b\leq c^2_{\alpha}(b_0)} T(b,\cdot)<\Phi^{-1}(\alpha)\}}(\cdot),$$
$$\varphi_{b_0}^{=}(\cdot) = \max\left(1_{\{T(b_0,\cdot)<\Phi^{-1}(\frac{\alpha}{2})\}}(\cdot), 1_{\{T(c^2_{\frac{\alpha}{2}}(b_0),\cdot)<\Phi^{-1}(\frac{\alpha}{2})\}}(\cdot)\right)$$

are consistent tests with asymptotic level α for $H_0 : b \geq b_0$, $H_0 : b \leq b_0$ and $H_0 : b = b_0$, respectively. In Denecke (2010) it is also shown that the assumption that a_0 is known is not necessary in case of testing $H_0 : b \geq b_0$. Figure 3 on the left provides the simulated power of the tests for $H_0 : b \geq 1$, for $N = 100$ uncontaminated data and 1000 repetitions each, here again in comparison to the test based on the classical maximum likelihood estimator, see Rinne (2009). The behavior of the power function for $H_0 : b \leq b_0$ is very similar to the behavior of the power function of the tests for the scale, see Figure 2. The right hand side of Figure 3 shows the behavior of the power functions of both tests in contaminated data. Here we contaminated each sample with 10% of the data coming from a Weibull distribution with shape $a_1 = 0.2$ and scale $b_1 = 5$. More pictures for different situations can be found in Denecke (2010).

Remark 2. Also consistent tests for censored data can be derived by this method. These tests have a better power than the tests based on the method of medians of He and Fung (1999) for high amount of censoring. If both parameters of the Weibull distribution are unknown a plug-in test can be used, see Denecke (2010).

Example 7 (Gumbel copula)**.** For the parameter of the Gumbel copula the correction $c_\alpha^1(\theta)$ seems to be $c_\alpha^1(\theta) = 2\theta$ and the conditions (11)-(16) seem to be satisfied, although we have no formal proof for it. For more details see Denecke (2010) and Denecke and Müller (2011). Using this c_α^1 we correct the test for the hypotheses $H_0 : \theta \geq \theta_0$ and $H_0 : \theta = \theta_0$. The simulated power of the corrected test for $H_0 : \theta \geq 2$ for $N = 100$ data and 1000 repetitions each is displayed in Figure 4 in comparison to the power function of the uncorrected test. We see that the correction improved very much the power of the test.

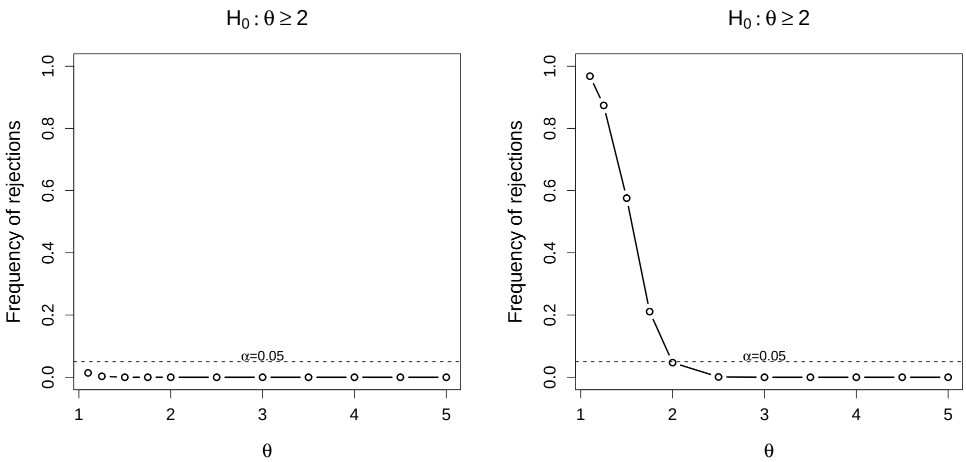

Figure 4: Power of the old (on the left) and the improved (on the right) test for $\theta \geq 2$.

Example 8 (Gaussian copula)**.** As a last example we consider tests for the parameter of the Gaussian copula. Here the conditions (11)-(16) are not fulfilled for all parameters, especially the function $p_{\rho,(\cdot)}$ is not monotone. But they are fulfilled for subsets of the parameter space so that the consistency of tests seems to be valid. Simulation studies also show, that the power of the corrected test based on likelihood depth is quite good and that the new test is also robust. As (11)-(14) are not fulfilled, we can not use the solution of $1 - p_{\rho,\rho} = p_{\rho_0,\rho}$ to determine $c_\alpha^1(\rho_0)$. Therefore, we estimated the value of $c_\alpha^1(\rho_0)$ for

ρ_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\hat{c}^1_{\alpha=0.05}(\rho_0)$	0.831	0.862	0.880	0.897	0.917	0.935	0.948	0.959	0.983

Table 1: Values of $c^1_{\alpha=0.05}(\rho_0)$.

different ρ_0 via simulation. Some of the results can be found in Table 1. Figure 5 shows the power function of this test for the hypothesis $H_0 : \rho = 0.7$ in uncontaminated and in contaminated data. Thereby the new test is compared with the "Fisher-Samiuddin-test" (Fisher-test), which is based on the test-statistic $\hat{t}(Z) = \frac{r(Z)-\rho_0\sqrt{N-2}}{\sqrt{(1-r^2(Z))(1-\rho_0^2)}}$ with $r(z) = \frac{\sum_{n=1}^N (x_n-\bar{x})(y_n-\bar{y})}{\sqrt{\sum_{n=1}^N (x_n-\bar{x})^2 \sum_{n=1}^N (y_n-\bar{y})^2}}$, where $\hat{t}(Z)$ has an asymptotic t_{N-2} distribution (Samiuddin 1970). We considered $N = 100$ data each and contaminated with 10% of the data coming from a two-dimensional normal distribution with correlation $\rho_1 = 0.05$. Figure 5 shows

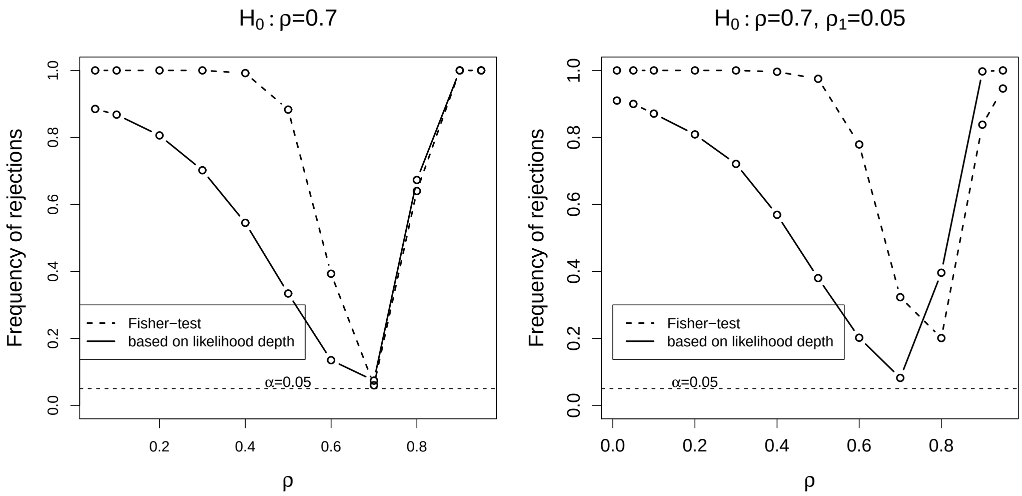

Figure 5: Test for the correlation coefficient, right hand-side: uncontaminated data, left hand-side: contaminated data.

that the test based on likelihood depth has still a worse power for $\rho < \rho_0$ compared to the Fisher-Samiuddin-test, but that it is robust against contamination with independent data in contrast to the classical test.

Discussion

In this paper, tests and estimators are developed for the one-parameter case. But sometimes, as in case of the Weibull distribution, the linking of two one-parametric problems can allow also the use in case of multi-dimensional problems. In general, when p-dimensional parameters are considered, the bias correction is not a map from $\mathbb{R}$ to $\mathbb{R}$, but from $\mathbb{R}^p$ to $\mathbb{R}^p$. So the correction can be determined, but the evaluation will be more complicated.

Appendix

Proof of Theorem 1: W.l.o.g., assume

$$\mathscr{C} := \max\left\{\limsup_{\theta\downarrow\inf\Theta} \sharp\{n;\ z_n \in T^\theta_{pos}\}, \limsup_{\theta\uparrow\sup\Theta} \sharp\{n;\ z_n \in T^\theta_{neg}\}\right\}.$$

Then there exists $\theta_1, \theta_2 \in \text{int}(\Theta)$ so that $\sharp\{n;\ z_n \in T^\theta_{pos}\} \leq \mathscr{C}$ for all $\theta < \theta_1$ and $\sharp\{n;\ z_n \in T^\theta_{neg}\} \leq \mathscr{C}$ for all $\theta > \theta_2$. This means

$$N \cdot d_T(\theta, z_*) = min(\sharp\{n;\ z_n \in T^\theta_{pos}\}, \sharp\{n;\ z_n \in T^\theta_{neg}\}) \leq \mathscr{C}$$

for all $\theta \notin [\theta_1, \theta_2]$. If $\overline{z}_1, \ldots, \overline{z}_N$ are observations where at most $M < \left\lfloor \frac{N+1}{2} - \mathscr{C} \right\rfloor$ of the original observations are replaced by other values, then $\sharp\{n;\ \overline{z}_n \in T^\theta_{pos}\} \leq \mathscr{C} + M < \left\lfloor \frac{N+1}{2} \right\rfloor$ for all $\theta < \theta_1$ and $\sharp\{n;\ \overline{z}_n \in T^\theta_{neg}\} \leq \mathscr{C} + M < \left\lfloor \frac{N+1}{2} \right\rfloor$ for all $\theta > \theta_2$. Hence $N \cdot d_T(\theta, \overline{z}_*) < \left\lfloor \frac{N+1}{2} \right\rfloor$ for all $\theta \notin [\theta_1, \theta_2]$. Since any parameter θ with maximum depth must satisfy $N \cdot d_T(\theta, \overline{z}_*) \geq \left\lfloor \frac{N+1}{2} \right\rfloor$, it holds $\tilde{\theta}_N(\overline{z}_*) \in [\theta_1, \theta_2] \subset \text{int}(\Theta)$ for the MD estimator at any corrupted sample $\overline{z}_* \in \mathcal{Y}_M(z_*)$. Hence breakdown can only happen if at least $\left\lfloor \frac{N+1}{2} - \mathscr{C} \right\rfloor$ observations are corrupted. □

Lemma 2. *For $Z_1, \ldots, Z_N$ i.i.d., $Z_i \sim F_{\theta'}$, $i = 1, \ldots, N$, and θ'' such that $p_{\theta',\theta''} = P_{\theta'}(T^{\theta''}_{pos}) \neq \frac{1}{2}$, it holds*

$$\sqrt{N}\, \frac{d_S(\theta'', Z_{*,N}) - 2p_{\theta',\theta''}(1 - p_{\theta',\theta''})}{2\sqrt{p_{\theta',\theta''}(1 - p_{\theta',\theta''})(1 - 2p_{\theta',\theta''})^2}} \xrightarrow{\mathscr{D}} X \sim \mathcal{N}(0, 1).$$

Proof: It holds

$$P_{\theta'}(d_T(\theta'', Z_* = (Z_1, Z_2)) = 1|Z_1 = z_1) = (1 - p_{\theta',\theta''})1_{T^{\theta''}_{pos}}(z_1) + p_{\theta',\theta''}1_{T^{\theta''}_{neg}}(z_1) \neq \tfrac{1}{2}$$

with probability one. To show that $\sqrt{N}\frac{d_S(\theta'', Z_{*,N}) - 2p_{\theta',\theta''}(1 - p_{\theta',\theta''})}{2\sqrt{p_{\theta',\theta''}(1 - p_{\theta',\theta''})(1 - 2p_{\theta',\theta''})^2}}$ is asymptotically normal distributed, the theorem of Hoeffding is used. As already mentioned, the simplicial depth is a U-statistic with tangent depth as kernel. In this situation the emergent quantities are:

$$\psi_{\theta''}(z_1, z_2) \ := \ 1_{\{d_T(\theta'', z_* = (z_1, z_2)) = 1\}}(z_1, z_2) = d_T(\theta'', z_* = (z_1, z_2)),$$

$$
\begin{aligned}
\gamma_{\theta',\theta''} &:= \mathrm{E}_{\theta'}(\psi_{\theta''}(Z_1,Z_2)) = \mathrm{E}_{\theta'}(1_{T^{\theta''}_{pos}}(Z_1)1_{T^{\theta''}_{neg}}(Z_2) + 1_{T^{\theta''}_{neg}}(Z_1)1_{T^{\theta''}_{pos}}(Z_2)) \\
&= 2p_{\theta',\theta''}(1-p_{\theta',\theta''}), \\
\psi_1(z_1) &:= \mathrm{E}_{\theta'}(\psi_{\theta',\theta''}(Z_1,Z_2)|Z_1=z_1) = (1-p_{\theta',\theta''})1_{T^{\theta''}_{pos}}(z_1) + p_{\theta',\theta''}1_{T^{\theta''}_{neg}}(z_1), \\
&\text{and} \\
\sigma^2_{\theta',\theta''} &:= \mathrm{Var}(\psi_1(Z_1)) = \mathrm{Var}((1-p_{\theta',\theta''})1_{T^{\theta''}_{pos}}(Z_1) + p_{\theta',\theta''}1_{T^{\theta''}_{neg}}(Z_1)) \\
&= (1-p_{\theta',\theta''})p_{\theta',\theta''}(1-2p_{\theta',\theta''})^2.
\end{aligned}
$$

The requirements of the theorem of Hoeffding are fulfilled as the U-statistic is not degenerated, because $\psi_1(z_1)$ is not independent of z_1. We get

$$
\sqrt{N}\frac{1}{\binom{N}{2}} \sum_{1\le n1<n2\le N} (d_T(\theta'',(Z_{n1},Z_{n2})) - \gamma_{\theta',\theta''}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4\sigma^2_{\theta',\theta''})
$$

under $P_{\theta'}$. □

Corollary 1. *For $p_{\theta,\theta} \neq \frac{1}{2}$, it holds $T(\theta,z_*) := \sqrt{N}\ \frac{d_S(\theta,Z_{*,N})-2p_{\theta,\theta}(1-p_{\theta,\theta})}{2\sqrt{(1-p_{\theta,\theta})p_{\theta,\theta}(1-2p_{\theta,\theta})^2}} \xrightarrow{\mathscr{D}} X \sim \mathcal{N}(0,1)$.*

Lemma 3. *Let $\frac{1}{2} < p \le \frac{1}{2} + \frac{1}{\sqrt{8}} \approx 0.85$.*

(a) If $1-p \le q \le p$, then $q(1-q) \ge p(1-p)$ and $q(1-q)(1-2q)^2 \le p(1-p)(1-2p)^2$.

(b) If $q \notin [1-p,p]$, then $q(1-q) < p(1-p)$.

Under the conditions of the last lemma we can determine c^1_α and $\check{c}^1_\alpha$, respectively c^2_α and $\check{c}^2_\alpha$.

Lemma 4. *If $p_{(\cdot),\theta_0} = P_{(\cdot)}(T^{\theta_0}_{pos})$ is strictly increasing from 0 to 1, $p_{\theta_0,(\cdot)}$ strictly decreasing and $\frac{1}{2} < p_{\theta_0,\theta_0} \le \frac{1}{2} + \frac{1}{\sqrt{8}}$, then for $\alpha < 0.5$, $c^1_\alpha(\theta_0)$ is the value $\tilde{\theta}$, such that $1 - p_{\tilde{\theta},\tilde{\theta}} = p_{\theta_0,\tilde{\theta}}$, and $\check{c}^1_\alpha(\theta_0)$ is the value $\tilde{\theta}$, such that $1 - p_{\theta_0,\theta_0} = p_{\tilde{\theta},\theta_0}$. In particular, $c^1_\alpha(\check{c}^1_\alpha(\theta_0)) = \theta_0 = \check{c}^1_\alpha(c^1_\alpha(\theta_0))$.*

Proof: We show only the proof for c^1_α since the proof for $\check{c}^1_\alpha(\theta_0)$ follows analogously. At first we transform $P_{\theta_0}(T(\theta,Z_*) < \Phi^{-1}(\alpha))$ such that Lemma 2 can be used, i.e.

$$
\begin{aligned}
P_{\theta_0}(T(\theta,Z_*) < \Phi^{-1}(\alpha)) = P_{\theta_0}\Bigg(& \sqrt{N}\frac{d_S(\theta,Z_*) - 2p_{\theta_0,\theta}(1-p_{\theta_0,\theta})}{2\sqrt{p_{\theta_0,\theta}(1-p_{\theta_0,\theta})(1-2p_{\theta_0,\theta})^2}} \\
& < \frac{\Phi^{-1}(\alpha)\sqrt{p_{\theta,\theta}(1-p_{\theta,\theta})(1-2p_{\theta,\theta})^2}}{\sqrt{p_{\theta_0,\theta}(1-p_{\theta_0,\theta})(1-2p_{\theta_0,\theta})^2}} + \sqrt{N}\frac{p_{\theta,\theta}(1-p_{\theta,\theta}) - p_{\theta_0,\theta}(1-p_{\theta_0,\theta})}{\sqrt{p_{\theta_0,\theta}(1-p_{\theta_0,\theta})(1-2p_{\theta_0,\theta})^2}}\Bigg).
\end{aligned}
$$

Since for θ it should hold $\lim_{N\to\infty} P_{\theta_0}(T(\theta, Z_*) < \Phi^{-1}(\alpha)) < \alpha$, ensure that

$$\frac{\Phi^{-1}(\alpha)\sqrt{p_{\theta,\theta}(1-p_{\theta,\theta})(1-2p_{\theta,\theta})^2}}{\sqrt{p_{\theta_0,\theta}(1-p_{\theta_0,\theta})(1-2p_{\theta_0,\theta})^2}} + \sqrt{N}\frac{p_{\theta,\theta}(1-p_{\theta,\theta}) - p_{\theta_0,\theta}(1-p_{\theta_0,\theta})}{\sqrt{p_{\theta_0,\theta}(1-p_{\theta_0,\theta})(1-2p_{\theta_0,\theta})^2}} \not\to \infty$$

as $N \to \infty$: Hence, $\frac{p_{\theta,\theta}(1-p_{\theta,\theta})-p_{\theta_0,\theta}(1-p_{\theta_0,\theta})}{\sqrt{p_{\theta_0,\theta}(1-p_{\theta_0,\theta})(1-2p_{\theta_0,\theta})^2}}$ should be smaller than or equal to zero, i.e.

$$p_{\theta,\theta}(1-p_{\theta,\theta}) - p_{\theta_0,\theta}(1-p_{\theta_0,\theta}) \leq 0.$$

Lemma 3 states that this is true, if and only if $1 - p_{\theta,\theta} \leq p_{\theta_0,\theta} \leq p_{\theta,\theta}$. Let be θ such that these inequalities hold, then using Lemma 2, we get

$$\begin{aligned} &P_{\theta_0}(T(\theta, Z_*) < \Phi^{-1}(\alpha)) \\ &\leq P_{\theta_0}\left(\sqrt{N}\frac{d_S(\theta, Z_*) - 2p_{\theta_0,\theta}(1-p_{\theta_0,\theta})}{2\sqrt{p_{\theta_0,\theta}(1-p_{\theta_0,\theta})(1-2p_{\theta_0,\theta})^2}} < \frac{\Phi^{-1}(\alpha)\sqrt{p_{\theta,\theta}(1-p_{\theta,\theta})(1-2p_{\theta,\theta})^2}}{\sqrt{p_{\theta_0,\theta}(1-p_{\theta_0,\theta})(1-2p_{\theta_0,\theta})^2}}\right) \\ &\overset{N\to\infty}{\to} \Phi\left(\frac{\Phi^{-1}(\alpha)\sqrt{p_{\theta,\theta}(1-p_{\theta,\theta})(1-2p_{\theta,\theta})^2}}{\sqrt{p_{\theta_0,\theta}(1-p_{\theta_0,\theta})(1-2p_{\theta_0,\theta})^2}}\right). \end{aligned}$$

As $\Phi^{-1}(\alpha) < 0$ for $\alpha < 0.5$ and since Lemma 3 yields $\frac{\sqrt{p_{\theta,\theta}(1-p_{\theta,\theta})(1-2p_{\theta,\theta})^2}}{\sqrt{p_{\theta_0,\theta}(1-p_{\theta_0,\theta})(1-2p_{\theta_0,\theta})^2}} \geq 1$, it is $\Phi\left(\frac{\Phi^{-1}(\alpha)\sqrt{p_{\theta,\theta}(1-p_{\theta,\theta})(1-2p_{\theta,\theta})^2}}{\sqrt{p_{\theta_0,\theta}(1-p_{\theta_0,\theta})(1-2p_{\theta_0,\theta})^2}}\right) \leq \alpha$. As $c^1_\alpha(\theta_0)$ is the maximum value θ for that these considerations are true, and $p_{\theta_0,(\cdot)}$ is decreasing, $c^1_\alpha(\theta_0)$ is the maximum value θ, such that $1 - p_{\theta,\theta} \leq p_{\theta_0,\theta} \leq p_{\theta,\theta}$, thus the solution of $p_{\theta_0,\theta} = 1 - p_\theta$ for θ. □

Corollary 2. *Under the assumptions of Lemma 4, $c^1_\alpha(\cdot)$ is strictly increasing if and only if $\check{c}^1_\alpha(\cdot)$ is strictly increasing.*

Lemma 5. *If c^i_α and $\check{c}^i_\alpha$, $i = 1, 2$, exist for every $\theta \in I$, $I \subset \Theta$ an interval, $c^i_\alpha(\check{c}^i_\alpha(\theta)) = \theta$ and are strictly monotone functions, then $c^i_\alpha(\cdot)$ and $\check{c}^i_\alpha(\cdot)$, $i = 1, 2$, are continuous on I.*

Proof: We write c_α and $\check{c}_\alpha$ instead of c^i_α and $\check{c}^i_\alpha$, $i = 1, 2$, because the proof is the same in both cases. W.l.o.g., assume c_α and $\check{c}_\alpha$ to be strictly increasing. Assume further c_α being not continuous. Using the fact that $c_\alpha(\check{c}_\alpha(\theta)) = \theta$ for all $\theta \in I$ then leads to contradiction.The proof that $\check{c}_\alpha$ is continuous works the same way. □

Proof of Theorem 2: (a) For $\theta < \check{c}^1_\alpha(\theta_0)$ we prove

$$\lim_{N\to\infty} P_\theta\left(\sup_{\tilde{\theta}\geq\theta_0} T(\tilde{\theta}, Z_{*,N}) \geq \Phi^{-1}(\alpha)\right) = 0,$$

where we use that for all z_*

$$\sup_{\tilde\theta\geq\theta_0} T(\tilde\theta, z_*) = \sup_{\tilde\theta\in\{([\theta_0,\infty)\cap\Theta)\cap\mathbb{Q}\}\cup\{\theta_0\}} T(\tilde\theta, z_*). \tag{17}$$

Assume that $\sup_{\tilde\theta\geq\theta_0} T(\tilde\theta, z_*) > \sup_{\tilde\theta\in\{([\theta_0,\infty)\cap\Theta)\cap\mathbb{Q}\cup\{\theta_0\}\}} T(\tilde\theta, z_*)$. Then there exists $\overline{\theta} \in ([\theta_0,\infty)\cap\Theta)\setminus\mathbb{Q}$ with

$$T(\overline{\theta}, z_*) = \sup_{\tilde\theta\geq\theta_0} T(\tilde\theta, z_*).$$

As $\frac{\partial}{\partial\theta}\ln L(\cdot, z)$ is continuous and $d_S(\cdot, z_*)$ a step function, there exists $\tilde\theta \in \mathbb{Q}$ near $\overline{\theta}$, such that $d_S(\overline{\theta}, z_*) = d_S(\tilde\theta, z_*)$. Further we have p_θ being continuous and so we get for every $\varepsilon > 0$ there exists $\tilde\theta$ near $\overline{\theta}$ such that $|T(\tilde\theta, z_*) - T(\overline{\theta}, z_*)| < \varepsilon$, this contradicts our assumption and consequently (17) holds. Let be $\theta < \check{c}^1_\alpha(\theta_0)$. Then

$$\begin{aligned}&\lim_{N\to\infty} P_\theta\left(\varphi^{0,\geq}_{\theta_0}(Z_{*,N}) = 0\right) = \lim_{N\to\infty} P_\theta\left(\sup_{\tilde\theta\geq\theta_0} T(\tilde\theta, Z_{*,N}) \geq \Phi^{-1}(\alpha)\right)\\ &\leq \sum_{\tilde\theta\in\{([\theta_0,\infty)\cap\Theta)\cap\mathbb{Q}\}\cup\{\theta_0\}} \lim_{N\to\infty} \underbrace{P_\theta(T(\tilde\theta, Z_{*,N}) \geq \Phi^{-1}(\alpha))}_{=0,\text{ see Theorem 4}} = 0\end{aligned}$$

which yields

$$\lim_{N\to\infty} P_\theta\left(\varphi^{0,\geq}_{\theta_0}(Z_{*,N}) = 1\right) = 1 - \lim_{N\to\infty} P_\theta\left(\varphi^{0,\geq}(Z_{*,N}) = 0\right) \geq 1.$$

For $\theta = \check{c}^1_\alpha(\theta_0)$ we apply the definition of $\check{c}^1_\alpha$ and get

$$\begin{aligned}&\lim_{N\to\infty} P_\theta\left(\varphi^{0,\geq}_{\theta_0}(Z_{*,N}) = 1\right) = \lim_{N\to\infty} P_\theta\left(\sup_{\tilde\theta\geq\theta_0} T(\tilde\theta, Z_{N,*}) < \Phi^{-1}(\alpha)\right)\\ &\leq \lim_{N\to\infty} P_\theta\left(T(\theta_0, Z_{*,N}) < \Phi^{-1}(\alpha)\right) \leq \alpha.\end{aligned}$$

Now consider $\theta > \check{c}^1_\alpha(\theta_0)$. As $\check{c}^1_\alpha$ is strictly increasing, we find $\overline{\theta} \geq \theta_0$ such that $\check{c}^1_\alpha(\overline{\theta}) < \theta < \overline{\theta}$, where the continuity of $\check{c}^1_\alpha$ for $\theta \in \Theta_0$, see Lemma 5, is used. Thus, with Theorem 4 yields

$$\lim_{N\to\infty} P_\theta\left(\sup_{\tilde\theta\geq\theta_0} T(\tilde\theta, Z_{*,N}) < \Phi^{-1}(\alpha)\right) \leq \lim_{N\to\infty} P_\theta\left(T(\overline{\theta}, Z_{*,N}) < \Phi^{-1}(\alpha)\right) = 0.$$

(b) We use the same arguments as above, as in the proof of Theorem 3, and the fact that $\check{c}^1_\alpha(c^1_\alpha(\theta_0)) = \theta_0$. □

Proof of Theorem 3: Let be $\theta < \theta_0$. As $c^1_\alpha(\cdot)$ is increasing, $\check{c}^1_\alpha(\cdot)$ is also increasing. Hence, we find with the help of Lemma 5 $\overline{\theta} \leq \theta_0$ such that $\check{c}^1_\alpha(\overline{\theta}) < \theta < \overline{\theta}$ as $\check{c}^1_\alpha(\tilde\theta) < \tilde\theta$ for all $\tilde{\theta}$. Then we know with Theorem 4 that $\lim_{N\to\infty} P_\theta\left(T(\overline{\theta}, Z_{*,N}) < \Phi^{-1}(\alpha)\right) = 0$, i.e. it holds

$$\lim_{N\to\infty} P_\theta\left(\sup_{\tilde{\theta}\leq\theta_0} T(\tilde{\theta}, Z_{*,N}) < \Phi^{-1}(\alpha)\right) \leq \lim_{N\to\infty} P_\theta\left(T(\overline{\theta}, Z_{*,N}) < \Phi^{-1}(\alpha)\right) = 0.$$

For $\theta = \theta_0$ it holds $\lim_{N\to\infty} P_\theta\left(\varphi_{\theta_0}^{0,\leq}(Z_{*,N}) = 1\right) \leq \alpha$. Now let be $\theta > \theta_0$. We prove $\lim_{N\to\infty} P_\theta\left(\sup_{\tilde{\theta}\leq\theta_0} T(\tilde{\theta}, Z_{*,N}) \geq \Phi^{-1}(\alpha)\right) = 0$, where we use that for all z_*

$$\sup_{\tilde{\theta}\leq\theta_0} T(\tilde{\theta}, z_*) = \sup_{\tilde{\theta}\in\{((-\infty,\theta_0]\cap\Theta)\cap\mathbb{Q}\}\cup\{\theta_0\}} T(\tilde{\theta}, z_*).$$

The proof for this works analogously to the proof of (17). Now let $\theta > \theta_0$. Then

$$\begin{aligned}
&\lim_{N\to\infty} P_\theta\left(\varphi_{\theta_0}^{\leq}(Z_{*,N}) = 0\right) = \lim_{N\to\infty} P_\theta\left(\sup_{\tilde{\theta}\leq\theta_0} T(\tilde{\theta}, Z_{*,N}) \geq \Phi^{-1}(\alpha)\right) \\
&= \lim_{N\to\infty} P_\theta\left(\sup_{\tilde{\theta}\in\{((-\infty,\theta_0]\cap\Theta)\cap\mathbb{Q}\}\cup\{\theta_0\}} T(\tilde{\theta}, z_*) \geq \Phi^{-1}(\alpha)\right) \\
&= \lim_{N\to\infty} P_\theta\left(\bigcup_{\tilde{\theta}\in\{((-\infty,\theta_0]\cap\Theta)\cap\mathbb{Q}\}\cup\{\theta_0\}} \{\tilde{\theta}; T(\tilde{\theta}, Z_{*,N}) \geq \Phi^{-1}(\alpha)\}\right) \\
&\leq \sum_{\tilde{\theta}\in\{((-\infty,\theta_0]\cap\Theta)\cap\mathbb{Q}\}\cup\{\theta_0\}} \lim_{N\to\infty} \underbrace{P_\theta(T(\tilde{\theta}, Z_{*,N}) \geq \Phi^{-1}(\alpha))}_{=0,\text{ see Theorem 4}} = 0.
\end{aligned}$$

Which leads directly to

$$\lim_{N\to\infty} P_\theta\left(\varphi_{\theta_0}^{0,\leq}(Z_{*,N}) = 1\right) = 1 - \lim_{N\to\infty} P_\theta\left(\varphi^{0,\leq}(Z_{*,N}) = 0\right) \geq 1.\square$$

Proof of Theorem 4:

(a) According to its definition, $\check{c}_\alpha^1(\theta_0)$ is the smallest value θ with

$$\lim_{N\to\infty} P_\theta\left(\varphi_{\theta_0}^{0,=}(Z_{*,N}) = 1\right) \leq \alpha$$

If $\theta < \check{c}_\alpha^1(\theta_0)$ or $\theta > \theta_0$, the proof of Lemma 4 shows $p_{\theta,\theta_0} \notin [1 - p_{\theta_0}, p_{\theta_0}]$ as $p_{(\cdot),\theta_0}$ is strictly increasing, hence, it is with a glance at Lemma 3 (b)

$$p_{\theta,\theta_0}(1 - p_{\theta,\theta_0}) < p_{\theta_0,\theta_0}(1 - p_{\theta_0,\theta_0}).$$

Using the proof of Lemma 4 again we get for these θ

$$\lim_{N\to\infty} P_\theta(\varphi_{\theta_0}^{0,=}(Z_{*,N}) = 1) = \lim_{N\to\infty} P_\theta(T(\theta_0, Z_{*,N}) < \Phi^{-1}(\alpha))$$

$$= \lim_{N\to\infty} P_\theta \left(\sqrt{N}\frac{d_S(\theta_0, Z_{*,N}) - 2p_{\theta,\theta_0}(1-p_{\theta,\theta_0})}{2\sqrt{p_{\theta,\theta_0}(1-p_{\theta,\theta_0})(1-2p_{\theta,\theta_0})^2}} \right.$$

$$\left. < \frac{\Phi^{-1}(\alpha)\sqrt{p_{\theta_0,\theta_0}(1-p_{\theta_0,\theta_0})(1-2p_{\theta_0,\theta_0})^2}}{\sqrt{p_{\theta,\theta_0}(1-p_{\theta,\theta_0})(1-2p_{\theta,\theta_0})^2}} + \sqrt{N}\underbrace{\frac{p_{\theta_0,\theta_0}(1-p_{\theta_0,\theta_0}) - p_{\theta,\theta_0}(1-p_{\theta,\theta_0})}{\sqrt{p_{\theta,\theta_0}(1-p_{\theta,\theta_0})(1-2p_{\theta,\theta_0})^2}}}_{>0} \right)$$

$$\underset{N\to\infty}{\longrightarrow} \Phi(\infty) = 1.$$

If $\check{c}^1_\alpha(\theta_0) < \theta < \theta_0$, it is $p_{\theta_0,\theta_0}(1-p_{\theta_0,\theta_0}) < p_{\theta,\theta_0}(1-p_{\theta,\theta_0})$ what leads to

$$\sqrt{N}\frac{p_{\theta_0,\theta_0}(1-p_{\theta_0,\theta_0}) - p_{\theta,\theta_0}(1-p_{\theta,\theta_0})}{\sqrt{p_{\theta,\theta_0}(1-p_{\theta,\theta_0})(1-2p_{\theta,\theta_0})^2}} \underset{N\to\infty}{\longrightarrow} -\infty,$$

i.e. $\lim_{N\to\infty} P_\theta\left(T(\theta_0, Z_{*,N}) < \Phi^{-1}(\alpha)\right) = 0$. If $\theta = \check{c}^1_\alpha(\theta_0)$, then $1 - p_{\theta_0,\theta_0} = p_{\theta,\theta_0}$ thus, $\frac{(1-2p_{\theta_0,\theta_0})^2}{(1-p_{\theta,\theta_0})^2} = \frac{(1-2p_{\theta_0,\theta_0})^2}{(2p_{\theta_0,\theta_0}-1)^2} = 1$. Consequently, it holds for $\theta = \check{c}^1_\alpha(\theta_0)$ and for $\theta = \theta_0$:

$$\lim_{N\to\infty} P_\theta(\varphi^{0,=}_{\theta_0}(Z_{*,N}) = 1) = \Phi(\Phi^{-1}(\alpha)) = \alpha.$$

(b) According to Corollary 1 and the definition of $c^1_\alpha(\theta_0)$ we have

$$\lim_{N\to\infty} P_{\theta_0}\left(\varphi^{=}_{\theta_0}(Z_{*,N}) = 1\right)$$
$$\leq \lim_{N\to\infty} P_{\theta_0}\left(T(\theta_0, Z_{*,N}) < \Phi^{-1}\left(\frac{\alpha}{2}\right)\right) + \lim_{N\to\infty} P_{\theta_0}\left(T(c^1_{\frac{\alpha}{2}}(\theta_0), Z_{*,N}) < \Phi^{-1}\left(\frac{\alpha}{2}\right)\right) = \alpha.$$

Now let be $\theta \neq \theta_0, \theta \in \Theta$. Using the results of (a) and the fact that $\check{c}^1_{\frac{\alpha}{2}}(c^1_{\frac{\alpha}{2}}(\theta_0)) = \theta_0$, see Lemma 4, leads to the following lines:

$$\lim_{N\to\infty} P_\theta\left(\varphi^{=}_{\theta_0}(Z_{*,N}) = 0\right)$$
$$= \lim_{N\to\infty} P_\theta\left(T(\theta_0, Z_{*,N}) \geq \Phi^{-1}\left(\frac{\alpha}{2}\right), T\left(c^1_{\frac{\alpha}{2}}(\theta_0), Z_{*,N}\right) \geq \Phi^{-1}\left(\frac{\alpha}{2}\right)\right)$$
$$\leq \min\{\lim_{N\to\infty}(1 - P_\theta\left(T(\theta_0, Z_{*,N}) < \Phi^{-1}\left(\frac{\alpha}{2}\right)\right)),$$
$$\lim_{N\to\infty}(1 - P_\theta\left(T\left(c^1_{\frac{\alpha}{2}}(\theta_0), Z_{*,N}\right) < \Phi^{-1}\left(\frac{\alpha}{2}\right)\right))\}$$
$$= \begin{cases} 0, & \theta < \check{c}^1_{\frac{\alpha}{2}}(\theta_0) \\ \min\{\lim_N(1 - P_\theta\left(T(\theta_0, Z_{*,N}) < \Phi^{-1}\left(\frac{\alpha}{2}\right)\right)), 0\} = 0, & \theta = \check{c}^1_{\frac{\alpha}{2}}(\theta_0) \\ \min\{1, 0\} = 0, & \check{c}^1_{\frac{\alpha}{2}}(\theta_0) < \theta < \theta_0 \\ \min\{0, 1\} = 0, & \theta_0 < \theta < c^1_{\frac{\alpha}{2}}(\theta_0) \\ \min\{0, \lim_N(1 - P_\theta\left(T\left(c^1_{\frac{\alpha}{2}}(\theta_0), Z_{*,N}\right) < \Phi^{-1}\left(\frac{\alpha}{2}\right)\right))\} = 0, & \theta = c^1_{\frac{\alpha}{2}}(\theta_0) \\ 0, & \theta > c^1_{\frac{\alpha}{2}}(\theta_0) \end{cases}.$$

Thus, we have for $\theta \neq \theta_0$

$$\lim_{N\to\infty} P_\theta\left(\varphi^{=}_{\theta_0}(Z_{*,N}) = 1\right) = 1 - \lim_{N\to\infty} P_\theta\left(\varphi^{=}_{\theta_0}(Z_{*,N}) = 0\right) \geq 1 - 0 = 1,$$

what proves the claim. □

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