Robust Estimators and Tests for Copulas based on Likelihood Depth

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Abstract

Estimators and tests based on likelihood depth for one-parametric copulas are given. For the Gaussian and Gumbel copula, it is shown that the maximum depth estimators are biased. They can be corrected and the new estimators are robust against contamination. For testing, the simplicial likelihood depth is considered. Because of the bias of the maximum depth estimator, the simplicial likelihood depth is not a degenerated U-statistic so that easily asymptotic α-level tests can be derived for arbitrary hypotheses. Tests are in particular investigated for $H_0 : \theta \leq \theta_0$ and $H_0 : \theta \geq \theta_0$. Simulation studies for the Gaussian and Gumbel copula show that the power of the first test is rather good, the latter one has to be improved, what is also done here. The new tests are robust against contamination.

Keywords: Copula, Gaussian copula, Gumbel copula, data depth, likelihood depth, simplicial depth, parametric estimation, test, robustness against contamination
AMS Subject Classification: Primary 62H20, 62H12, 62H15; Secondary 62H10, 62F03, 62F12

1 Introduction

The copula model has a variety of applications because it models dependence structures. For example in finance, in the analysis of credit risks, the insolvency of several debtors at the same time or for insurances the risk of appearance of different claims at the same time have to be modeled to insure solvency of the bank and insurance, respectively, all the time. Copulas can also be used in the simulation of technical production processes to model the occurrence of coupled failures. Some applications of copulas can be found in Aas (2004), Andresen (2005), Cízek, Härdle and Weron (2005), Dobríc and Schmid

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(2005) or Malvergne and Sornette (2006). For an introduction to copulas see for example Joe (1997) or Nelsen (2006).

Different estimation procedures for copulas were introduced. Parametric, semi-parametric and nonparametric methods are proposed. Most of the parametric and semi-parametric methods are two-stage estimations, as presented in Andresen (2005), Genest, Ghoudi and Rivest (1995), Hoff (2007) or Kim, Silvapulle and Silvapulle (2007) for example. Here in most cases as a first step the margins are estimated by parametric or non-parametric methods, then an estimation procedure for the parameter of the copula is presented. An example for a nonparametric estimation model for the copula is the empirical copula, see Durrleman, Nikeghbali and Roncalli (2000) or Capéraà, Fougères and Genest (1997). Goodness-of-fit-tests can be found e.g. in Dobríc and Schmid (2005), Fermian (2005) or Panchenko (2005).

In this work, we derive estimators and tests for one-parametric two-dimensional copulas via likelihood depth and simplicial likelihood depth. Likelihood depth and simplicial likelihood depth are rather general notions of data depth, at first used by Mizera and Müller (2004) and Müller (2005). They extend the half space depth of Tukey (1975) and the simplicial depth of Liu (1988,1990) which lead to outlier robust generalizations of the median for multivariate data. They belong to a broad class of depth notions introduced and studied in the last 20 years, see e.g. Rousseeuw and Hubert (1999), Zuo and Serfling (2000a,b), Mizera (2002), and the book of Mosler (2002). Although likelihood depth bases on a parametric approach, it can lead to distribution-free estimators and tests as Mizera and Müller (2004) demonstrated for location-scale estimation and Müller (2005) for regression. Müller (2005) also showed that simplicial likelihood depth is in particular appropriate for testing since it is an U-statistic. Thereby rather general hypotheses can be tested and the resulting tests are outlier robust.

Copulas are often given by distributional assumptions on the form of the copula. This distributional assumptions for the copula will be used here to define likelihood depth and simplicial likelihood depth for copulas. The approach is demonstrated for the Gaussian copula and the Gumbel copula for two dimensions which are based on one parameter only. However, the approach can also be used for other one-parametric copulas.

In Section 2, the basic concepts of likelihood depth are given and specified for the case of one unknown parameter $\theta \in \Theta \subset \mathbb{R}$. Moreover, the definitions of Gaussian copula and Gumbel copula are given. Section 3 provides the main results for estimating the parameter θ of a Gaussian copula and Gumbel copula via likelihood depth and simplicial likelihood depth. The resulting estimators are biased but can be corrected. They are robust against contamination.

Tests for general hypotheses about the parameter θ are derived in Section 4.1 via simplicial likelihood depth. Since the maximum depth estimator is biased, the simplicial likelihood depth is not a degenerated U-statistic as this is the case for most simplicial depth notions. Hence its asymptotic distribution is simply the normal distribution so that asymptotic α-level tests can be derived easily. Simulation studies show that these tests have a reasonable power for testing $H_0 : \theta \leq \theta_0$ for the Gaussian copula and the Gumbel copula. In particular, the test for the Gaussian copula parameter ρ is as powerful as the classical Fisher-Samiuddin test. But the power is bad for testing $H_0 : \theta \geq \theta_0$ because of the bias of the underlying estimator. Therefore an improvement of the tests is proposed which leads to rather powerful tests. All new tests show also high robustness against contamination.

2 Preliminaries

2.1 Likelihood depth

Let $Z_1, \ldots Z_N$ be i.i.d. with density f_θ, $\theta \in \Theta \subset \mathbb{R}^q$. The likelihood function at parameter θ and observation z_n will be denoted by $L(\theta, z_n) := f_\theta(z_n)$. Now we are able to define the global likelihood depth similar to Mizera (2002), Müller (2005):

Definition 2.1. *The* global likelihood depth *of a parameter θ within observations $z_1, \ldots, z_N$ is the minimal number m of $z_{i_1}, \ldots, z_{i_m}$, such that θ is a likelihood nonfit within $\{z_1, \ldots, z_N\} \setminus \{z_{i_1}, \ldots, z_{i_m}\}$, which means, one can find $\theta' \neq \theta$ such that $L(\theta', z_n) > L(\theta, z_n)$ for every $z_n \in \{z_1, \ldots, z_N\} \setminus \{z_{i_1}, \ldots, z_{i_m}\}$.*

In large datasets the calculation of the global likelihood depth can be complicated. Mizera (2002) defined the tangent likelihood depth and Müller (2005) introduced the simplicial likelihood depth, which are easier to handle.

Definition 2.2.

(i) The tangent likelihood depth *of θ within $z_* := (z_1, \ldots, z_N)^T$ is*

$$d_T(\theta, z_*) := \frac{1}{N} \inf_{u \neq 0} \sharp\{n; u^T h_n'(\theta) \leq 0\}.$$

Where $h_n(\theta) := \ln(L(\theta, z_n))$ and $h_n'(\theta)$ is the vector of the partial derivatives of $h_n(\theta)$ for $\theta = (\theta_1, \ldots, \theta_q)$ (especially for $\theta \in \mathbb{R}$, $h_n'(\theta) = \frac{\partial}{\partial\theta} \ln f_\theta(z_n)$).

(ii) The simplicial likelihood depth *of θ within observations $z_* := (z_1, \ldots, z_N)^T$ is defined as*

$$d_S(\theta, z_*) := \binom{N}{q+1}^{-1} \sharp\{\{n_1, \ldots, n_{q+1}\} \subset \{1, \ldots, N\}; d_T(\theta, (z_{n_1}, \ldots, z_{n_{q+1}})) > 0\},$$

where q is the dimension of θ.

An estimator $\hat{\theta}$ for the parameter θ can be chosen as the one in the parameter-space Θ that has maximum depth, i.e. $\theta \in \arg\max d_i(\theta, z_*)$, $i = T, S$.

We are going to treat especially one-dimensional parameter θ. In this case the depths are calculated by counting the observations z_n, $n \in \{1, \ldots, N\}$, for which $h_n'(\theta) = \frac{\partial}{\partial\theta} \ln f_\theta(z_n)$ is positive, negative and zero respectively. These numbers will be denoted by $N_{pos}^\theta = \sharp\{n; \frac{\partial}{\partial\theta} \ln f_\theta(z_n) > 0\}$, $N_{neg}^\theta = \sharp\{n; \frac{\partial}{\partial\theta} \ln f_\theta(z_n) < 0\}$ and $N_0^\theta = \sharp\{n; \frac{\partial}{\partial\theta} \ln f_\theta(z_n) = 0\}$.

Lemma 2.3. *The tangent likelihood depth of* $\theta \in \mathbb{R}$ *in data* $z_* = (z_1, \ldots, z_N)^T$, $z_i \in \mathbb{R}^2, i = 1, \ldots, N$, *is*

$$d_T(\theta, z_*) = \frac{1}{N}(\min(N_{pos}^\theta, N_{neg}^\theta) + N_0^\theta).$$

The simplicial likelihood depth is figured by calculating the tangent depth of each pair of observations z_{i_1}, z_{i_2}, $i_1, i_2 \in \{1, \ldots, N\}, i_1 \neq i_2$. The tangent depth is only non-zero if and only if $h'_{i_1}(\theta)h'_{i_2}(\theta) \leq 0$.

Lemma 2.4. *The simplicial likelihood depth of* $\theta \in \mathbb{R}$ *in data* $z_* = (z_1, \ldots, z_N)^T$, $z_i \in \mathbb{R}^2, i = 1, \ldots, N$, *is*

$$d_S(\theta, z_*) = \frac{2}{N(N-1)} \left(N_{pos}^\theta N_{neg}^\theta + N_{pos}^\theta N_0^\theta + N_{neg}^\theta N_0^\theta + \binom{N_0^\theta}{2} \right).$$

Asymptotically it is $N_0^\theta \approx 0$, then $d_S(\theta, z_*) = \frac{2}{N(N-1)} N_{pos}^\theta N_{neg}^\theta$. For simplification we consider $N_0^\theta = 0$ and N to be even in this work. Consequently both depths reach their maximum when N_{pos}^θ equals N_{neg}^θ.

2.2 Copulas

Any two dimensional distribution function H of random variables X and Y can be expressed by the copula and the distribution functions F and G of the marginal distributions, i.e.

$$H(x, y) = C(F(x), G(y)) \quad \text{or} \quad C(u, v) = H(F^{-1}(u), G^{-1}(v)).$$

Hence the copula $C : [0, 1]^2 \to [0, 1]$ describes the dependence structure between X and Y.

If (X, Y) has a two-dimensional normal distribution, where X and Y have standard normal distribution, then the copula is called Gaussian copula. Since the correlation coefficient ρ describes completely the dependence structure between X and Y, the parameter of the Gaussian copula is the correlation coefficient ρ.

Definition 2.5 (see e.g. Aas (2004))**.** *The* Gaussian copula *is defined as*

$$C_\rho(u, v) := \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \exp\left(\frac{-(s^2 - 2\rho st + t^2)}{2(1-\rho^2)} \right) ds\, dt,$$

$u, v \in (0, 1), -1 < \rho < 1$, *where* Φ *denotes the one-dimensional standard normal distribution function.*

As a second example we examine a one-parametric family of Archimedean copulas, the *Gumbel* copulas.

Definition 2.6 (see Nelsen (2006))**.** *Let* $\theta \geq 1$ *and* $\varphi_\theta : [0,1] \to [0,\infty]$, $t \mapsto \varphi_\theta(t) := (-\ln t)^\theta$. *Then the copula* $C_\theta : [0,1] \times [0,1] \to [0,1]$ *given by*

$$C_\theta(u,v) = \exp\left(-((-\ln u)^\theta + (-\ln v)^\theta)^{\frac{1}{\theta}}\right)$$

is called Gumbel copula*.*

A special property of the Gumbel copula is the concentration of points in $(1,1)$ and $(0,0)$.

We will denote with θ the parameter of the Gumbel copula as well as the parameter of an arbitrary copula. Data (u_n, v_n), $n = 1, \ldots, N$, from a Gaussian copula or a Gumbel copula can be generated with the function `rcopula` of the R-package "copula", see Yan (2007). Data from Gaussian copula can be also generated via data (x_n, y_n), $n = 1, \ldots, N$, from a two-dimensional normal distribution with marginal distributions given by the standard normal distribution, with the help of the function `rmnorm` of the same package. Setting $(u_n, v_n) = (\Phi(x_n), \Phi(y_n))$ leads then to data from the Gaussian copula.

3 Estimation

3.1 Maximum depth estimator

Assume, we have a dataset $z_* = (z_1, \ldots, z_N)^T$, $z_n = (u_n, v_n), n = 1, \ldots, N$, and know the corresponding copula family except for the parameter. Using the tangent likelihood depth we define an estimator for the unknown parameter as that parameter $\theta \in \Theta$ with maximum tangent likelihood depth. The same can be done using the simplicial likelihood depth. To calculate these depths, the density function f_θ and thus the likelihood function as

$$L(\theta, z) = f_\theta(z) = \frac{\partial}{\partial u}\frac{\partial}{\partial v} C_\theta(u,v),$$

must be calculated in a first step. In a second step, $h_n'(\theta) = \frac{\partial}{\partial \theta} \ln L(\theta, z_n) = \frac{\partial}{\partial \theta} \ln f_\theta(u_n, v_n)$ has to be determined. These derivatives are given for the Gaussian copula and the Gumbel copula by the following lemma.

Lemma 3.1. *It holds*

$$\frac{\partial}{\partial \rho} \ln f_\rho(u,v) = \frac{-\rho y^2 + (1+\rho^2)\, x\, y + \rho - \rho^3 - \rho x^2}{(1-\rho^2)^2} \, \frac{1}{\Phi'(x)\; \Phi'(y)}$$

with $x = \Phi^{-1}(u)$ *and* $y = \Phi^{-1}(v)$ *for the Gaussian copula and*

$$\begin{aligned} \frac{\partial}{\partial \theta} \ln f_\theta(u,v) \;&=\; \ln x + \ln y + \left((x^\theta + y^\theta)^{\frac{1}{\theta}} - 1\right) \frac{1}{\theta^2} \ln(x^\theta + y^\theta) \\ &\quad + \left(\frac{1}{\theta} - \frac{(x^\theta + y^\theta)^{\frac{1}{\theta}}}{\theta} - 2\right) \frac{x^\theta \ln x + y^\theta \ln y}{x^\theta + y^\theta} \\ &\quad + \frac{1 + (x^\theta + y^\theta)^{\frac{1}{\theta}} \left(-\frac{\ln(x^\theta + y^\theta)}{\theta^2} + \frac{x^\theta \ln(x) + y^\theta \ln(y)}{\theta(x^\theta + y^\theta)}\right)}{\theta - 1 + (x^\theta + y^\theta)^{\frac{1}{\theta}}} \end{aligned}$$

with $x := -\ln u$ *and* $y := -\ln v$ *for the Gumbel copula.*

Proof: We use for the Gumbel copula that

$$f_\theta(z) = \frac{(-\ln v)^{\theta-1}(-\ln u)^{\theta-1}}{uv} e^{-((-\ln u)^\theta+(-\ln v)^\theta)^{\frac{1}{\theta}}} \left(\theta - 1 + ((-\ln u)^\theta + (-\ln v)^\theta)^{\frac{1}{\theta}}\right)((-\ln u)^\theta + (-\ln v)^\theta)^{\frac{1}{\theta}-2}$$

for $z = (u, v) \in [0, 1] \times [0, 1]$, $\theta \geq 1$. □

Hence for calculating the tangent likelihood depth and the simplicial tangent depth of a parameter θ according to Lemma 2.3 and 2.4, it must be only counted how many observations z_n satisfy $\frac{\partial}{\partial\theta} \ln f_\theta(z_n) > 0$, $\frac{\partial}{\partial\theta} \ln f_\theta(z_n)) < 0$, and $\frac{\partial}{\partial\theta} \ln f_\theta(z_n) = 0$. The parameter with maximum depth, i.e. the maximum depth estimator $\hat{\theta}$, was determined with the help of an interval search. For the Gaussian copula, the interval we started with is of course $[0, 1)$. For the Gumbel copula, the starting interval was determined as $[1, 10 \cdot MLE(\theta)]$, where $MLE(\theta)$ denotes the maximum likelihood estimator for θ.

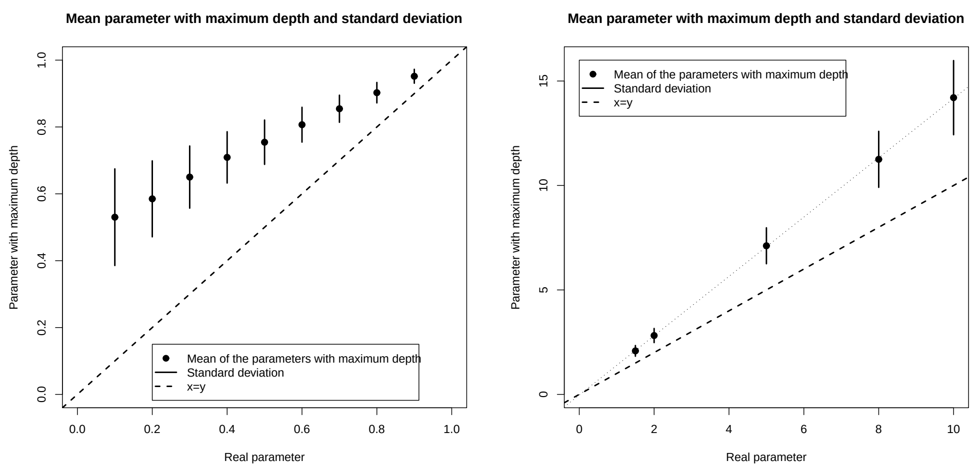

Figure 1: Mean and standard deviation of the simulated maximum depth estimators

Figure 1 shows the mean and the standard deviations of maximum tangent depth estimators of simulated data sets. Thereby 100 data points from the Gaussian copula and the Gumbel copula with different parameters were randomly generated. This was repeated 1000 times for each constellation. Figure 1 shows that the mean of the estimators $\hat{\rho}$ and $\hat{\theta}$, respectively, over the 1000 simulations, is always greater than the real parameter ρ and θ, respectively. This means that the estimator is biased.

Since for large data sets, the maximum depth for tangent likelihood depth as well as for simplicial likelihood depth is attained for θ with $N^\theta_{pos} \approx N^\theta_{neg}$, the maximum depth estimator based on tangent likelihood depth and the maximum depth estimator based on simplicial likelihood depth coincide in most cases. Hence the maximum simplicial depth estimator is also biased.

3.2 Asymptotic bias of the estimator

The bias also exists asymptotically, i.e. for $N \to \infty$. To calculate the asymptotic bias, set

$$T_\theta^{pos} := \{z = (u,v); \frac{\partial}{\partial\theta} \ln f_\theta(u,v) > 0\},\ T_\theta^{neg} := \{z_n = (u_n, v_n); \frac{\partial}{\partial\theta} \ln f_\theta(u,v) < 0\}$$

and

$$p_{\theta,\theta'} := P_\theta(T_{\theta'}^{pos}) = \int\int 1_{T_{\theta'}^{pos}}(u,v) f_\theta(u,v)\, du\, dv = 1 - P_\theta(T_{\theta'}^{neg}).$$

The parameter θ' with maximum depth is the one with $N_{pos}^{\theta'} \approx N_{neg}^{\theta'}$. The law for large numbers provide

$$\frac{1}{N} N_{pos}^{\theta'} \overset{\theta}{\longrightarrow} p_{\theta,\theta'} \quad \text{and} \quad \frac{1}{N} N_{neg}^{\theta'} \overset{\theta}{\longrightarrow} 1 - p_{\theta,\theta'}$$

for $N \to \infty$, if θ is the underlying parameter. Hence a maximum depth estimator will be only asymptotically unbiased if $p_{\theta,\theta} = 1/2$ since only then $N_{pos}^\theta \approx N_{neg}^\theta$ holds asymptotically. However, $p_{\theta,\theta} = 1/2$ is not satisfied for the Gaussian copula and the Gumbel copula which can be seen from Figure 2. Figure 2 shows that, except for $\rho = 0$ for the Gaussian copula, it always holds $p_{\theta,\theta} > 1/2$. Hence, except for $\rho = 0$ for the Gaussian copula, there is an asymptotic bias. The exception for $\rho = 0$ can be seen easily from $\frac{\partial}{\partial\rho} \ln f_\rho(u,v)|_{\rho=0} = \frac{\Phi^{-1}(u)\, \Phi^{-1}(v)}{\Phi'(\Phi^{-1}(u))\, \Phi'(\Phi^{-1}(v))}$. Then we have $\frac{\partial}{\partial\rho} \ln f_\rho(u,v)|_{\rho=0} > 0$ if and only if $x = \Phi^{-1}(u)$ and $y = \Phi^{-1}(v)$ have the same sign and this holds under $\rho = 0$ with probability equal to 1/2.

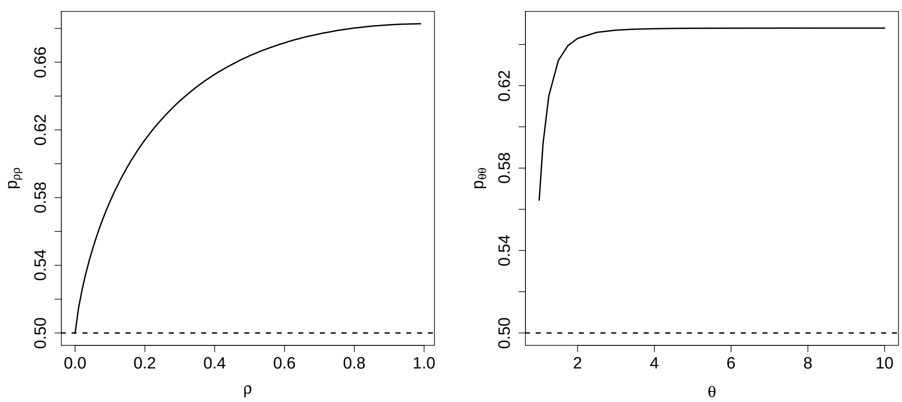

Figure 2: Plots of $(\rho, p_{\rho,\rho})$ for the Gaussian copula and $(\theta, p_{\theta,\theta})$ for the Gumbel copula.

The asymptotic bias can be determined by that parameter θ' such that $p_{\theta,\theta'} = 1/2$, since the maximum depth estimator will converge to that θ'. There are fixed relations between θ and θ' so that we set $b(\theta) = \theta'$. These relations are plotted in Figure 3. Note that the relation seems to be linear for the Gumbel copula.

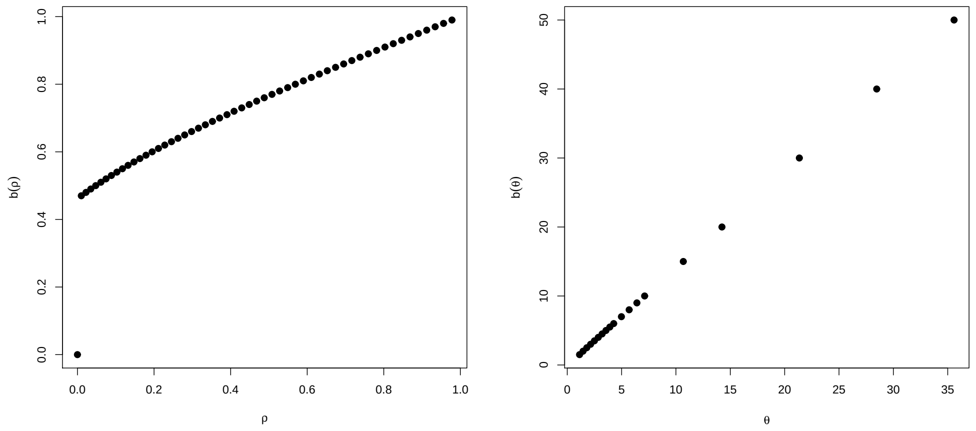

Figure 3: Relation between ρ and $b(\rho)$for Gaussian copula (left hand side) and θ and $b(\theta)$ for Gumbel copula (right hand side)

The relation between ρ and $b(\rho)$ for the parameter ρ of the Gaussian copula is more complicated. The relation is not linear as Figure 3 shows. Moreover, we have $b(0) = 0$ and $b(\rho) > \beta$ with $\beta \approx 0.461$ for $\rho > 0$. The reason for this jump shows Figure 4. It shows that for $\rho = 0$ there exists two parameter ρ', namely $\rho' = 0$ and $\rho' = 0.461$, with $p_{0,\rho'} = 1/2$. $p_{0,\rho}$ is larger than 1/2 for $\rho \in (0, 0.461)$ and then decreasing for $\rho > 0.461$. All numerical calculations showed that only for $\rho = 0$ there exists two solutions ρ' with $p_{0,\rho'} = 1/2$. But we have no proof for this. For small $\rho > 0$, like $\rho = 0.01$ in Figure 4, the function $p_{\rho,\cdot}$ changes only a little bit. But then we have $p_{\rho,0} > 1/2$ so that the solution ρ' with $p_{\rho,\rho'} = 1/2$ is unique implying that $b(\rho) = \rho'$ is well defined.

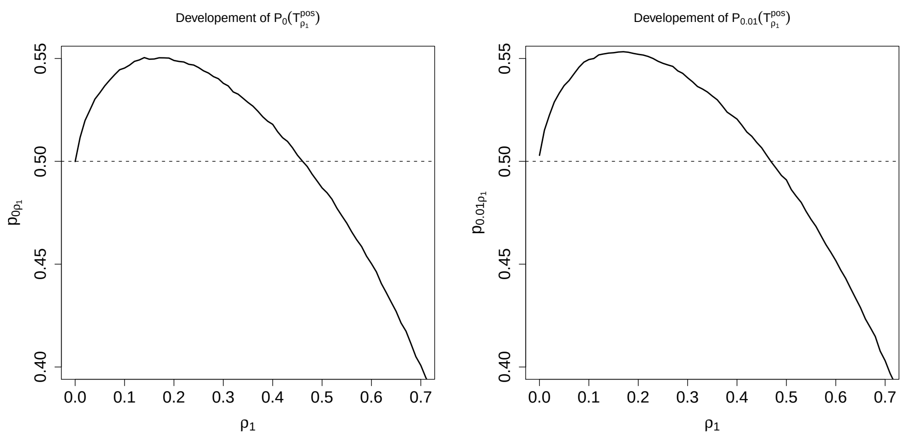

Figure 4: Developing of $p_{\rho,\rho_1} = P_\rho(T^{pos}_{\rho_1})$ for $\rho = 0$ and $\rho = 0.01$ for the Gaussian copula

The probability $p_{\theta,\theta'} = P_\theta(T^{pos}_{\theta'})$ was calculated by first interpolating the zeros $r(u) < s(u)$ of $\frac{\partial}{\partial\theta} \ln f_\theta(z)|_{\theta=\theta'}$ depending on u and then determining by numerical integration $p_{\theta,\theta'} = \int_0^1 \int_{r(u)}^{s(u)} f_\theta(u,v) dv du$. Thereby we used that for every u at most two zeros $r(u) < s(u)$

exist. In particular, for the Gaussian copula the zeros can be given explicitly. In case of the Gumbel copula, the points $(0,0)$ and $(1,1)$ are singular points of the density function, thus the evaluation of the integral is made with higher accuracy here.

Alternatively $p_{\theta,\theta'}$ could be calculated by generating a large number of points $z=(u,v)$ from the distribution with θ and then checking whether $\frac{\partial}{\partial\theta}\ln f_\theta(u,v)|_{\theta=\theta'}>0$ is satisfied.

For determining $b(\theta)$ with $P_\theta(T^{pos}_{b(\theta)})=0.5$, i.e. $|P_\theta(T^{pos}_{b(\theta)})-0.5|<10^{-4}$, the bisection method was used.

3.3 Unbiased estimator based on likelihood depth

Since

$$\arg\max_\theta d_T(\theta,z)\xrightarrow{\theta} b(\theta)$$

for $N\to\infty$, the bias can be corrected by using the inverse of the mapping b. Such inverse mapping a satisfies

$$a(\arg\max_\theta d_T(\theta,z))\xrightarrow{\theta} a(b(\theta))=\theta,$$

for $N\to\infty$ so that the transformed maximum depth estimator is unbiased.

Figure 3 shows that the relation between θ and $b(\theta)$ for the Gumbel copula is given by a line. We have no proof for this, but all points $(\theta,b(\theta))$, we calculated, are lying on a line. Hence the inverse mapping a is also linear and given by

$$a(\beta)=0.015+0.71\,\beta.$$

The unbiased estimator for the parameter of the Gumbel copula is therefor

$$\hat\theta(z)=0.015+0.71\ \arg\max_{\theta\geq 1} d_T(\theta,z).$$

Since the bias function b for the Gaussian copula has a jump at 0, the inverse mapping a is calculated here as

$$a(\beta)=\inf\{\rho\in[0,1];\ b(\rho)\geq\beta\}.$$

This means that $a(\beta)$ is set as zero for $\beta\in[0,0.461]$. A classical inverse mapping a of b exists for $\beta>0.461$. This inverse mapping a was estimated by a least square fit of a polynomial of degree three to the points $(b(\rho),\rho)$, for $b(\rho)=0.4615,0.47,0.48,\ldots,0.99$. This leads to

$$a(\beta)=-1.24101\,\beta^3+3.68702\,\beta^2-1.4546\,\beta+0.00857.$$

We decided for a polynomial of degree three, because the maximum absolute error for compensation with a polynomial of degree two was 0.006, for compensation with degree three 0.00041 and for degree four only little smaller (0.0004). Hence the new unbiased estimator for the correlation ρ can be set as

$$\begin{aligned}\hat{\rho}(z) &= -1.22217(\arg\max_{\rho} d_T(\rho, z_*))^3 + 3.6434(\arg\max_{\rho} d_T(\rho, z_*))^2 \\ &\quad -1.42154(\arg\max_{\rho} d_T(\rho, z_*)) + 0.000396,\end{aligned}$$

if $\arg\max_\rho d_T(\rho, z_*) > 0.461$ and $\hat{\rho}(z) = 0$ for $\arg\max_\rho d_T(\rho, z_*) \in [0, 0.461]$. If we have $\arg\max_\rho d_T(\rho, z_*) < 0$, then we set $\hat{\rho}(z) = -\hat{a}(|\arg\max_\rho d_T(\rho, z_*)|)$.

Note, for the Gumbel copula as well as for the Gaussian copula, it can happen that there is more than one parameter with maximum depth, $\tilde{\theta}_1, \tilde{\theta}_2 \ldots \in \arg\max_\theta d_T(\theta, z_*)$. Then the maximum depth estimator is calculated as the mean of the arguments $\tilde{\theta} := mean(\tilde{\theta}_1, \tilde{\theta}_2, \ldots)$ and the new unbiased estimator is $\hat{\theta} = a(\tilde{\theta})$.

3.4 Comparison with existing estimators

This new estimator for the Gumbel copula is compared with the Maximum Likelihood Estimator (MLE) for the Gumbel copula for datasets with different θ (1000 times, 100 data points each). Table 1 shows the real parameter (θ), the mean of the new estimator ($\hat{\theta}$) and the mean of the MLE (MLE).

θ	1.1	1.5	2.0	3.0	4.0	5.0	10.0
$\hat{\theta}$	1.050	1.497	2.041	3.030	4.050	5.048	10.104
MLE	1.137	1.504	2.020	3.023	4.031	5.028	10.109

Table 1: Comparison of $\hat{\theta}$ and MLE for the Gumbel copula

This indicates, that for simulated datasets from a Gumbel copula the new estimator is nearly as good as the MLE.

The new estimator for the Gaussian copula, i.e. for the correlation coefficient ρ, was compared with the correlation coefficient of Pearson-Bravais. For calculating the correlation coefficient of Pearson-Bravais the data were transformed by $(x_n, y_n) = (\Phi^{-1}(u_n), \Phi^{-1}(u_n))$ for $n = 1, \ldots, N$. Since the marginal distributions of the distribution providing (x_n, y_n) are standard normal distributions we tried also the mean of $x_n \cdot y_n$; but this provided larger mean squared errors.

Table 2 shows the means of the new estimator $\hat{\rho}$ in comparison to the means of the correlation coefficient of Pearson ("Ps").

ρ	0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
$\hat{\rho}$	0.065	0.127	0.204	0.300	0.395	0.491	0.587	0.690	0.799	0.897	0.978
Ps	0.006	0.063	0.201	0.298	0.397	0.497	0.595	0.696	0.798	0.899	0.990

Table 2: New estimator for ρ compared with Pearson's correlation coefficient Ps

This shows that Pearson's correlation coefficient is only in some cases slightly better than the new estimator.

3.5 Estimation for data with unknown margins

So far we assumed that the data are coming from a distribution, where both marginal distributions are uniform on the interval $[0, 1]$. In practice however, the marginal distributions must be estimated and the data have to be transformed so that the distribution on $[0, 1]$ of the transformed data is uniform. The marginal distributions F and G can be approximated by the cumulative empirical distribution function for example, the estimated functions shall be denoted by $\hat{F}$ and $\hat{G}$. Then the data are transformed to $\tilde{z}_n = (\hat{F}(x_n), \hat{G}(y_n)) \in [0, 1]$ for $n = 1, \ldots, N$. For this transformed dataset we can have a look at the belonging copula. First we have to decide for one copula family and then use the methods from above to estimate the parameter. Table 3 provides some results for the estimation as comparison of the MLE and the new estimator ($\hat{\theta}$), where we simulated data with different marginals and dependence structure given by the Gumbel copula with $\theta = 2$. We simulated 1000 datasets with 100 data each. t_k denotes the t-distribution with $df = k$, $\mathcal{N}(0, 1)$ the normal distribution with mean 0 and variance 1, $\mathcal{E}(2)$ the exponential distribution with rate 2, and χ^2_2 the χ^2-distribution with $df = 2$. Table 3 shows that the new estimator achieves most times slightly better results than the MLE. Similar results hold for other parameters and for the Gaussian copula.

F	G	$\hat{\theta}$	MLE
t_3	t_3	2.012	1.977
t_8	t_8	2.027	1.974
t_3	χ^2_2	2.019	1.987
$\mathcal{N}(0, 1)$	$\mathcal{N}(0, 1)$	2.016	1.974
$\mathcal{N}(0, 1)$	$\mathcal{E}(2)$	2.007	1.971

Table 3: Estimation for unknown marginal distributions

3.6 Robustness against contamination

To study robustness properties, we regard ϵ-contaminations of the underlying distribution P_{θ_1} with a distribution P_{θ_2}, i.e. the data are simulated from $(1-\epsilon)P_{\theta_1} + \epsilon P_{\theta_2}$, where $0 < \epsilon << 1/2$. We assume that both distributions, P_{θ_1} and P_{θ_2}, are coming from the same class of copulas and differ only in the parameter. Again 100 data points were simulated 1000 times for each constellation.

θ_1	1.1	2	5	10	10
θ_2	100	1.1	1.1	1.1	2
$\epsilon \cdot 100$	10	5	5	10	10
$\hat{\theta}$	1.25	1.97	4.87	9.02	9.16
MLE	37.78	1.94	4.28	5.47	7.14

Table 4: Comparison of estimators for the Gumbel parameter for data coming from a ϵ-contaminated distribution

Table 4 shows some results for the Gumbel copula. It demonstrates that the new estimator $\hat{\theta}$ is more robust than the Maximum Likelihood Estimator MLE. This holds for $\theta_1 < \theta_2$ as well as for $\theta_1 > \theta_2$. Moreover, while the MLE is breaking down for $\theta_2 \to \infty$, i.e. it goes to ∞ for $\theta_2 \to \infty$, the new estimator is stable.

ρ_1	ρ_2	$\epsilon \cdot 100$	$\hat{\rho}$	Ps	ρ_1	ρ_2	$\epsilon \cdot 100$	$\hat{\rho}$	Ps
0.10	0.01	10	0.114	0.093	0.01	0.99	10	0.228	0.107
0.30	0.01	10	0.273	0.271	0.20	0.99	10	0.362	0.275
0.50	0.01	10	0.456	0.448	0.40	0.99	10	0.521	0.456
0.70	0.01	10	0.662	0.630	0.60	0.99	10	0.685	0.636
0.99	0.01	10	0.978	0.891	0.90	0.99	10	0.918	0.907

Table 5: Estimation of the correlation parameter ρ for data coming from ϵ-contamination

Table 5 provides simulation results for the parameter ρ of the Gaussian copula. Here the new estimator $\hat{\rho}$ is only more robust than Pearson's correlation coefficient Ps if the ϵ-contamination is given by a parameter ρ_2 which is smaller than ρ_1. In the case $\rho_1 < \rho_2$, Pearson's correlation coefficient is more robust. But this changes completely if we contaminate a two-dimensional normal distribution with a completely different distribution, namely a distribution providing outliers at (x_0, x_0) for large x_0. I.e. data (x_n, y_n), $n = 1, \ldots, N$, were simulated from $(1-\epsilon)P_\rho + \epsilon\delta_{x_0,x_0}$, where δ_{x_0,x_0} is the Dirac measure on (x_0, x_0) and P_ρ is a two-dimensional normal distribution with mean $(0, 0)$ and covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. The new estimator $\hat{\rho}$ was based then on $(u_n, v_n) = (\hat{F}(x_n), \hat{G}(y_n))$ for $n = 1, \ldots, N$, where $\hat{F}$ and $\hat{G}$ are again the empirical distribution functions of the marginals. Table 6 shows that now the new estimator $\hat{\rho}$ is much more robust. In particular, there is no breakdown of the new estimator, while Pearson's correlation coefficient is breaking down, i.e. it reaches the upper bound of 1. Similar results hold for other parameters ρ and ε.

ρ	x_0	$\varepsilon \cdot 100$	$\hat{\rho}$	Ps
0.1	10	10	0.449	0.956
0.1	100	10	0.444	0.999
0.1	1 000	10	0.450	1
0.1	1 000 000	10	0.448	1

Table 6: Estimation of ρ for data from $(1-\epsilon)P_\rho + \epsilon\delta_{x_0,x_0}$

4 Tests

4.1 Tests based on simplicial depth

The tangent depth of θ in $\{z_1, z_2\}$ is according to Lemma 2.3 with probability one

$$d_T(\theta, z_* = (z_1, z_2)) = 1_{T_\theta^{pos}}(z_1) 1_{T_\theta^{neg}}(z_2) + 1_{T_\theta^{neg}}(z_1) 1_{T_\theta^{pos}}(z_2),$$

where 1_A shall denote the indicator function of A. The simplicial depth satisfies with probability one $d_S(\theta, z_*) = \frac{1}{\binom{N}{2}} \sum_{1 \le n_1 < n_2 \le N} d_T(\theta, (z_{n_1}, z_{n_2}))$, so it is a U-statistic belonging to the tangent depth. As soon as we have $p_{\theta,\theta} = P_\theta(T_\theta^{pos}) \neq \frac{1}{2}$, then

$$P_\theta \left(d_T(\theta, Z_* = (Z_1, Z_2)) = 1 | Z_1 = z_1 \right) = \left(1 - p_{\theta,\theta}\right) 1_{T_\theta^{pos}}(z_1) + p_{\theta,\theta} 1_{T_\theta^{neg}}(z_1) \not\equiv \tfrac{1}{2}$$

with probability 1. In such cases, $d_S(\theta, Z_*)$ is not a degenerated U-statistic and its asymptotic distribution is a normal distribution according to Hoeffding's theorem (see e.g. Lee 1990). Since $p_{\theta,\theta} \neq \frac{1}{2}$ for all parameters of the Gumbel copula and for all parameters $\rho \in (-1, 1) \setminus \{0\}$ of the Gaussian copula, easily asymptotic α-level tests can be obtained. Hence the bias of the maximum depth estimators allows a simple testing approach.

Theorem 4.1. *If $p_\theta := p_{\theta,\theta} = P_\theta(T_\theta^{pos}) \neq \frac{1}{2}$ for all $\theta \in \Theta_0$ and*

$$T(\theta, z_*) := \sqrt{N} \frac{\frac{2}{N(N-1)} \sum_{1 \le n_1 < n_2 \le N} d_T(\theta, (z_{n_1}, z_{n_2})) - 2p_\theta(1 - p_\theta)}{2\sqrt{(1 - p_\theta) p_\theta (1 - 2p_\theta)^2}},$$

then the test $\varphi(z_*) := 1_{\{\sup_{\theta \in \Theta_0} T(\theta, z) < \Phi^{-1}(\alpha)\}}(z_*)$ is an asymptotic α-level test for $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \notin \Theta_0$.

Proof: The emergent quantities in the theorem of Hoeffding (see e.g. Lee 1990) are:

$$\begin{aligned}
\psi_\theta(z_1, z_2) &:= 1_{\{d_T(\theta, z_* = (z_1, z_2)) = 1\}}(z_1, z_2) = d_T(\theta, z_* = (z_1, z_2)) \\
\gamma_\theta &:= \mathrm{E}(\psi_\theta(Z_1, Z_2)) = \mathrm{E}(1_{T_\theta^{pos}}(Z_1) 1_{T_\theta^{neg}}(Z_2) + 1_{T_\theta^{neg}}(Z_1) 1_{T_\theta^{pos}}(Z_2)) \\
&= 2p_\theta(1 - p_\theta) \\
\psi_1(z_1) &:= \mathrm{E}(\psi_\theta(Z_1, Z_2) | Z_1 = z_1) = (1 - p_\theta) 1_{T_\theta^{pos}}(z_1) + p_\theta 1_{T_\theta^{neg}}(z_1) \\
\sigma_\theta^2 &:= \mathrm{Var}(\psi_1(Z_1)) = \mathrm{Var}((1 - p_\theta) 1_{T_\theta^{pos}}(Z_1) + p_\theta 1_{T_\theta^{neg}}(Z_1)) \\
&= \mathrm{Var}((1 - p_\theta) 1_{T_\theta^{pos}}(Z_1)) + \mathrm{Var}(p_\theta 1_{T_\theta^{neg}}(Z_1)) \\
&\quad + 2\mathrm{Cov}((1 - p_\theta) 1_{T_\theta^{pos}}(Z_1), p_\theta 1_{T_\theta^{neg}}(Z_1)) \\
&= (1 - p_\theta)^3 p_\theta + p_\theta^3 (1 - p_\theta) + 2p_\theta(1 - p_\theta) [\overbrace{\mathrm{E}(1_{T_\theta^{pos}}(Z_1) 1_{T_\theta^{neg}}(Z_1))}^{=0} \\
&\quad - \mathrm{E}(1_{T_\theta^{pos}}(Z_1)) \mathrm{E}(1_{T_\theta^{neg}}(Z_1))] \\
&= (1 - p_\theta) p_\theta (1 - 2p_\theta)^2.
\end{aligned}$$

The requirements of the theorem of Hoeffding are fulfilled, and the U-statistic is not degenerated, because $\psi_1(z_1)$ is not independent of z_1. Hence we get

$$\sqrt{N}\frac{1}{\binom{N}{2}}\sum_{1\leq n1<n2\leq N}(d_T(\theta,(Z_{n1},Z_{n2}))-\gamma_\theta)\xrightarrow{\mathcal{L}}\mathcal{N}(0,4\sigma_\theta^2),$$

i.e. the test statistic defined as

$$T(\theta,z_*):=\sqrt{N}\frac{\frac{2}{N(N-1)}\sum_{1\leq n1<n2\leq N}d_T(\theta,(z_{n1},z_{n2}))-\gamma_\theta}{2\sigma_\theta}$$

has approximately a standard normal distribution. For $\theta'\in\Theta_0$, it is
$P^Z_{\theta'}(\varphi=1)=P_{\theta'}(\sup_{\theta\in\Theta_0}T(\theta,Z)<\Phi^{-1}(\alpha))\leq P_{\theta'}(T(\theta',Z)<\Phi^{-1}(\alpha))\overset{N\to\infty}{\to}\Phi(\Phi^{-1}(\alpha))=\alpha.$
Thus φ is an asymptotic α-level test. □

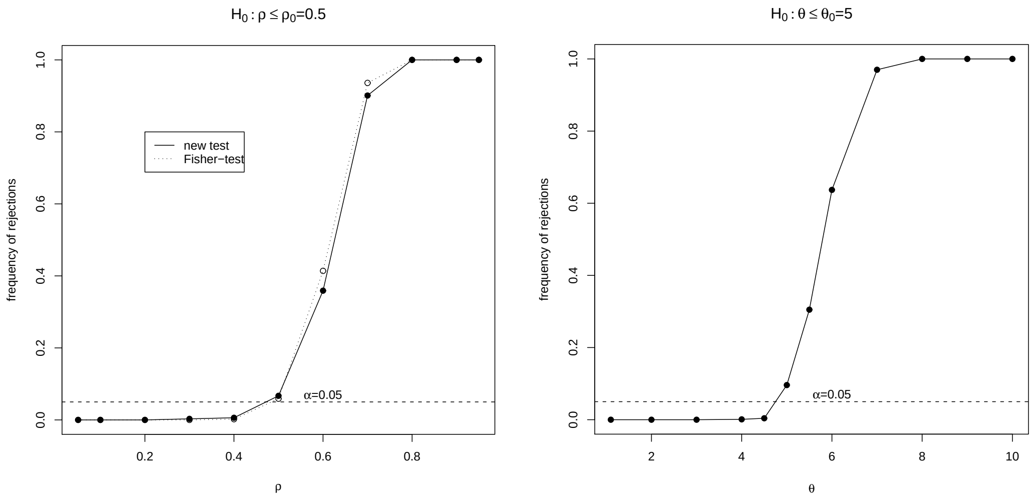

Figure 5: Power functions for sample size N=100 for the Gaussian copula (left hand side) and the Gumbel copula (right hand side)

In particular, asymptotic α-level tests for $H_0:\theta\leq\theta_0$ and $H_0:\theta\geq\theta_0$ can be obtained from Theorem 4.1. These tests keep approximately the level already for finite samples and are powerful in the case $H_0:\theta\leq\theta_0$, as can be seen from Figure 5. The power values were again generated by 1000 repetitions of $N=100$ data for each parameter. In the case of the Gaussian copula, the comparison of the test for $H_0:\rho\leq\rho_0$ with the classical Fisher-Samiuddin test (Samiuddin (1970)) shows that the test based on simplicial likelihood depth is as powerful as the Fisher-Samiuddin test (see the left hand side of Figure 5).

However, the power of tests for $H_0:\theta\geq\theta_0$ are very bad which can be seen for the Gaussian copula from the left hand side of Figure 6. This holds for other parameters of the Gaussian copula as well as for the Gumbel copula. It is caused by the bias of the maximum simplicial depth estimator. The problem is that the simplicial depth and hereby the test-statistic does in most cases not reach its maximum for the real parameter θ of the data but for a $\theta'>\theta$.

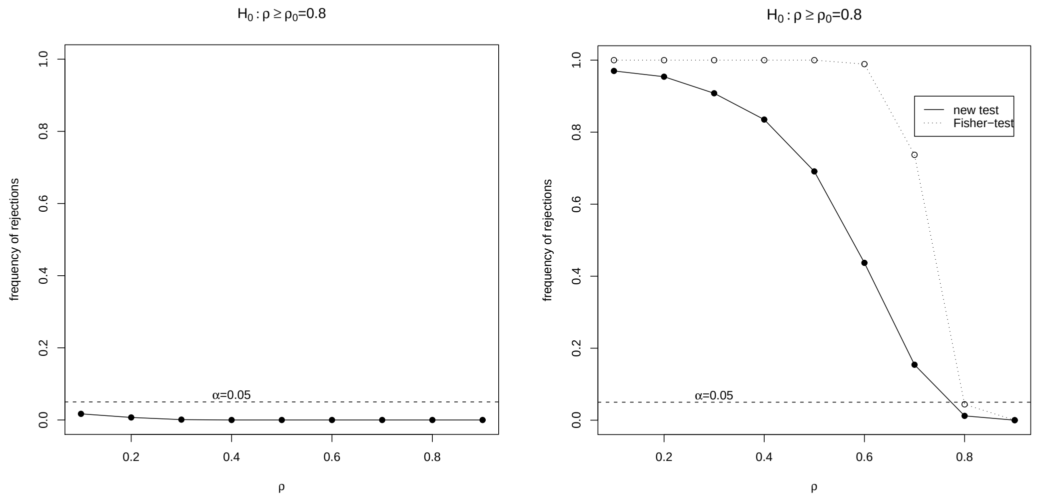

Figure 6: Power functions for sample size N=100 for the original test for the Gaussian copula (left hand side) and the improved test for the Gaussian copula (right hand side)

This means that often the test-statistic reaches the maximum for a $d(\theta) > \theta$, so there can be a θ with $\theta < \theta_0$ but $d(\theta) > \theta_0$. For this θ, $H_0 : \theta \geq \theta_0$ would falsely not be rejected. This leads to the bad power of the test.

4.2 Improvement of the power of the tests for $H_0 : \theta \geq \theta_0$

The idea to improve the power is to determine $c_\alpha(\theta_0)$ such that

$$c_\alpha(\theta_0) = \max\{\theta; \lim_{N\to\infty} P_{\theta_0}\left(T(\theta, Z_*) < \Phi^{-1}(\alpha)\right) \leq \alpha\}.$$

Theorem 4.2. *If $c_\alpha(\cdot)$ is strictly increasing, then the test given by*

$$\varphi(z_*) = 1_{\{\sup_{\theta \geq c_\alpha(\theta_0)} T(\theta,z) < \Phi^{-1}(\alpha)\}}(z_*)$$

is asymptotically an α-level test for $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$.

Proof: Let be $\theta_1 \geq \theta_0$, then we get

$$\begin{aligned} \lim_{N\to\infty} P_{\theta_1}\big(\sup_{\theta \geq c_\alpha(\theta_0)} T(\theta, Z_*) < \Phi^{-1}(\alpha)\big) &\leq \lim_{N\to\infty} P_{\theta_1}\big(\sup_{\theta \geq c_\alpha(\theta_1)} T(\theta, Z_*) < \Phi^{-1}(\alpha)\big) \\ &\leq \lim_{N\to\infty} P_{\theta_1}(T(c_\alpha(\theta_1), Z_*) < \Phi^{-1}(\alpha)) \leq \alpha. \quad \square \end{aligned}$$

ρ	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$\hat{c}_{\alpha=0.05}(\rho)$	0.819	0.831	0.852	0. 862	0.872	0.88	0.888	0.897	0.902
ρ	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9
$\hat{c}_{\alpha=0.05}(\rho)$	0.917	0.923	0.935	0.939	0.948	0.957	0.959	0.976	0.983

Table 7: Values of $c_{\alpha=0.05}(\rho)$ for the Gaussian copula

θ_0	1.25	1.5	2	2.5	3	3.5	4	4.5	5	6	7	8	9	10
$\hat{c}_{\alpha=0.05}(\theta_0)$	2.5	3	4	5	6	7	8	9	10	12	14	16	18	20

Table 8: Values of $c_{\alpha=0.05}(\theta)$ for the Gumbel copula

According to Theorem 4.2, the strict increase of $c_\alpha(\cdot)$ must be checked. Since $c_\alpha(\theta)$ can be only calculated numerically, this property can be checked only by examples. Table 7 shows some values for the Gaussian copula and Table 8 for the Gumbel copula. They indicate that the strict increase is satisfied. Moreover, we have the following proposal for the Gumbel copula.

Proposal 4.3. *It holds* $c_{\alpha=0.05}(\theta) = 2\theta$ *for the Gumbel copula.*

The right hand side of Figure 6 shows the power function for the improved test for the Gaussian test. Still the power is worse than the Fisher-Samiuddin test but it is now reasonable. The same holds for other parameters ρ_0 and for the Gumbel copula. For the Gumbel copula see also the left hand side of Figure 7.

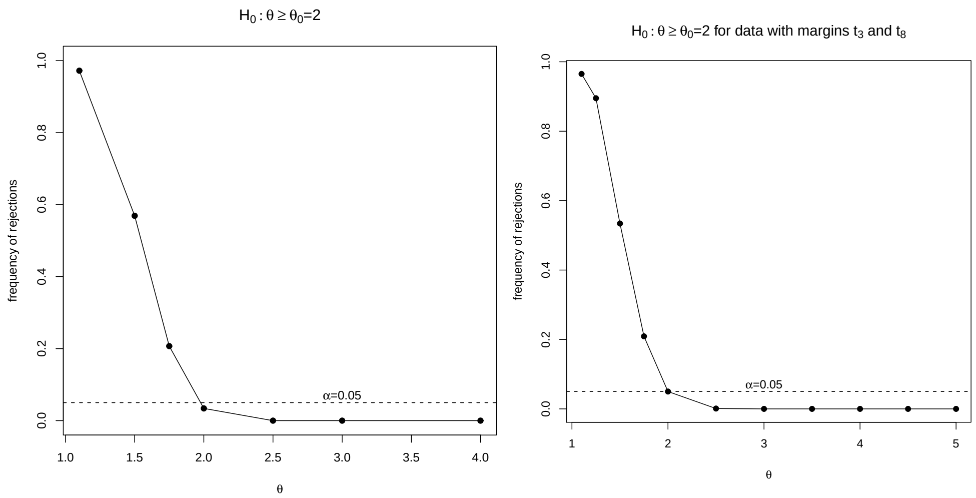

Figure 7: Power of tests for data from Gumbel copula with given margins (left hand side) and with estimated margins (right hand side)

4.3 Tests for data with unknown margins

As soon as the margins are unknown they must be estimated by the empirical distribution function $\hat{F}$ and $\hat{G}$. Then data given by $\tilde{z_n} = (\hat{F}(x_n), \hat{G}(y_n)) \in [0,1]$ for $n = 1, \ldots, N$ are used as in Section 3.5. The right hand side of Figure 7 shows the power function if the both marginal distributions were generated by a t_3 distribution and a t_8 distribution, respectively, and then estimated by $\hat{F}$ and $\hat{G}$. The comparison with the left hand side of this figure, where the marginal distributions were known, shows almost no difference. This holds also for other marginal distributions and for the Gaussian copula.

4.4 Robustness against contamination

As in Section 3.6, the ϵ-contamination $(1-\epsilon)P_\theta+\epsilon P_{\theta_2}$ is considered for the Gumbel copula. Figure 8 shows the power function for a test for $H_0 : \theta \leq 2$ for 10%-contamination with P_{θ_2} with $\theta_2 = 10,\ 10^6$. It shows that the level is biased but there is no breakdown of the level, i.e. the power at $\theta_0 = 2$ is not going to 1 with increasing θ_2. The same holds for other parameters θ_0, θ_2, and ϵ. A similar behavior holds also for testing $H_0 : \theta \geq \theta_0$. But here a breakdown cannot be expected since the alternative is bounded.

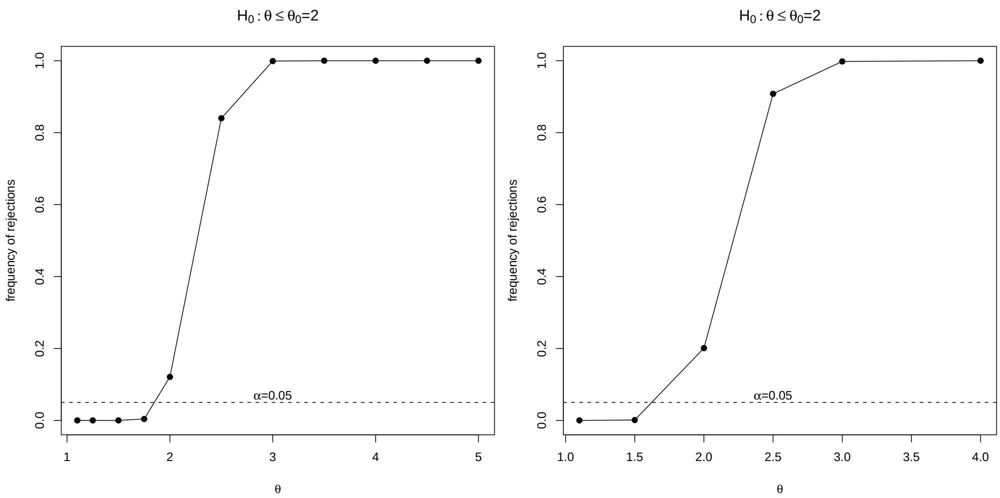

Figure 8: Power for Gumbel copula with outliers with increasing θ_2, $\theta_2 = 10, 10^6$

The behavior of the power functions for testing the correlation parameter ρ are shown in Figure 9. For testing e.g. $H_0 : \rho \geq \rho_0 = 0.8$, the improved test based on simplicial depth is robust against 10%-contamination with P_{ρ_2} with $\rho_2 = 0.01$, which means that the level is not strongly biased. This is in opposite to the Fisher-Samiuddin test which has a large bias of the level. See the left hand side of Figure 9. We obtained similar results for other ρ_0, ρ_2, and ϵ. However, the power functions for testing $H_0 : \rho \leq \rho_0$ in the presence of ϵ-contamination with P_{ρ_2} with $\rho_2 >> \rho_0$ are similar for the new test and the Fisher-Samiuddin test. In such situations the Fisher-Samiuddin test is quite robust. This corresponds to the results for estimation given in Section 3.6.

But if we compare the tests for $H_0 : \rho \leq \rho_0$ in the presence of ϵ-contamination of two-dimensional normal distributions with $\delta_{(x_0,x_0)}$ as considered in Section 3.6, then the new test is much more robust than the Fisher-Samiuddin test (see the right hand side of Figure 9). In particular the level of the new test is not breaking down for x_0 going to infinity which is the case for the Fisher-Samiuddin test.

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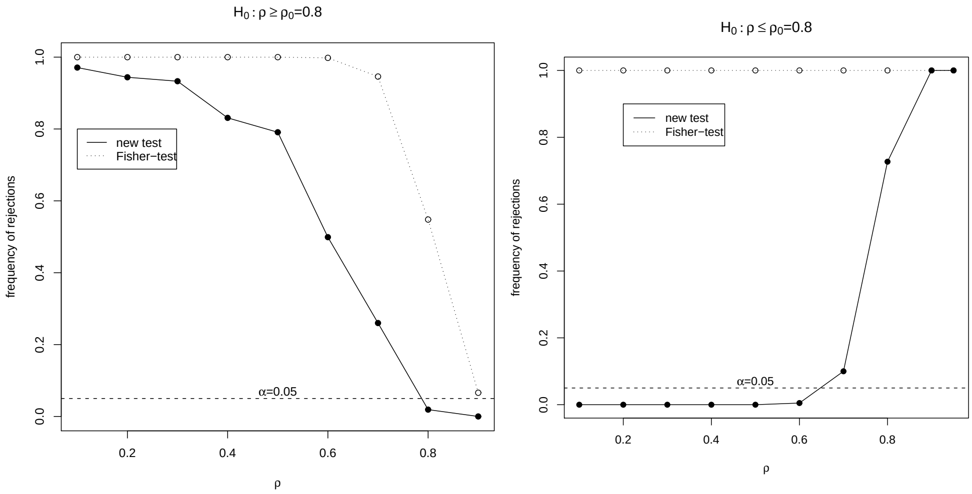

Figure 9: Power comparison of the tests for the correlation coefficient: with 10%-contamination with P_{ρ_2} with $\rho_2 = 0.01$ (left hand side) and 10%-contamination with $\delta_{(x_0,x_0)}$ with $x_0 = 100\,000$ (right hand side)

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